Linear Algebra
Solution Extra Exercise 0

G-17

19 IX 2025

The extra exercises are written by me and can contain errors. You should first solve exercises from the lecture material. These exercises are optional.

Exercise 1

Assume you can write all vectors in $\mathbb{R}^4$ as a linear combination of the vectors

$$\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \in \mathbb{R}^4$$

Does the following sets of vectors also have all the vectors in $\mathbb{R}^4$ as a linear combination?

a) $\{10\mathbf{v}_1, 3\mathbf{v}_2, \mathbf{v}_3, 2\mathbf{v}_4\}$

b) $\{2\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 - 2\mathbf{v}_2, \mathbf{v}_4, 7\mathbf{v}_1 - 4\mathbf{v}_2\}$

c) $\{\mathbf{v}_1 - \mathbf{v}_2, \mathbf{v}_2 + \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_1 + \mathbf{v}_4\}$

Hint:[1] If n vectors $\mathbf{v}_1, \dots, \mathbf{v}_n \in \mathbb{R}^n$ can generate all the vectors in $\mathbb{R}^n$ then two linear combinations with $\lambda_1, \dots, \lambda_n, \mu_1, \dots, \mu_n \in \mathbb{R}$

$$\lambda_1\mathbf{v}_1 + \cdots + \lambda_n\mathbf{v}_n \text{ and } \mu_1\mathbf{v}_1 + \cdots + \mu_n\mathbf{v}_n$$

are equal to each other if and only if

$$\lambda_1 = \mu_1, \lambda_2 = \mu_2, \dots, \lambda_n = \mu_n$$

.

[1] We will see why and also prove this hint in the following weeks.

Solution 1

We are going to develop some ideas during the course of the following weeks, like linear independence and the concept of bases, that will help us solve this question in a faster and more elegant way. However the idea that uses the knowledge of week 0 is as following: If you can write an arbitrary vector $\mathbf{v} \in \mathbb{R}^4$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$, this means that there exists unique [2] scalars $\lambda, \mu, \alpha, \gamma \in \mathbb{R}$ such that

$$\lambda \mathbf{v}_1 + \mu \mathbf{v}_2 + \alpha \mathbf{v}_3 + \gamma \mathbf{v}_4 = \mathbf{v}$$

Now we look for new coefficients to multiply our new vectors in the given sets to produce the same linear combination of vectors. If we can find the corresponding coefficients then we can proudly say that the given set of vectors can produce each vector in $\mathbb{R}^4$ and this is possible by multiplying the vectors with these coefficients. Don't forget that we already know $\lambda, \mu, \alpha, \gamma$ *i.e.* we already know how to generate a given vector $\mathbf{v} \in \mathbb{R}^4$ using $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$.

a)

$$a \cdot (10\mathbf{v}_1) + b \cdot (3\mathbf{v}_2) + c \cdot (\mathbf{v}_3) + d \cdot (2\mathbf{v}_4) = \lambda \mathbf{v}_1 + \mu \mathbf{v}_2 + \alpha \mathbf{v}_3 + \gamma \mathbf{v}_4$$

We can find the corresponding new coefficients $a, b, c, d \in \mathbb{R}$ using the hint:

$$\begin{aligned} 10a = \lambda &\Rightarrow a = \lambda/10 \\ 3b = \mu &\Rightarrow b = \mu/3 \\ c = \alpha &\Rightarrow c = \alpha \\ 2d = \gamma &\Rightarrow d = \gamma/2 \end{aligned}$$

This proves that it is possible to generate all the vectors in $\mathbb{R}^4$ using this set of vectors.

b)

$$a \cdot (2\mathbf{v}_1 + \mathbf{v}_2) + b \cdot (\mathbf{v}_1 - 2\mathbf{v}_2) + c \cdot (\mathbf{v}_4) + d \cdot (7\mathbf{v}_1 - 4\mathbf{v}_2) = \lambda \mathbf{v}_1 + \mu \mathbf{v}_2 + \alpha \mathbf{v}_3 + \gamma \mathbf{v}_4$$

We can formulate the left hand side as:

[2] We will prove why these scalars must be unique in the following weeks.

$$(2a + b + 7d) \cdot \mathbf{v}_1 + (a - 2b - 4d) \cdot \mathbf{v}_2 + 0 \cdot \mathbf{v}_3 + c \cdot \mathbf{v}_4$$

Now we can try to use the hint. However for $\mathbf{v}_3$ we have a coefficient 0 on the left hand side. So we have a constraint $0 = \alpha$. This means that using the given set of vectors, we cannot generate the linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ that use $\mathbf{v}_3$. For example we cannot generate the resulting vector from $4\mathbf{v}_1 + 12\mathbf{v}_2 + 2\mathbf{v}_3 + 1\mathbf{v}_4$ because to get to that vector we must use 2 of the vector $\mathbf{v}_3$.

So it is NOT possible to generate all vectors from $\mathbb{R}^4$ using this set of vectors.

c)

$a \cdot (\mathbf{v}_1 - \mathbf{v}_2) + b \cdot (\mathbf{v}_2 + \mathbf{v}_3) + c \cdot (\mathbf{v}_4) + d \cdot (\mathbf{v}_1 + \mathbf{v}_4) = \lambda \mathbf{v}_1 + \mu \mathbf{v}_2 + \alpha \mathbf{v}_3 + \gamma \mathbf{v}_4$

We can write the left hand side as:

$$(a + d) \cdot \mathbf{v}_1 + (b - a) \cdot \mathbf{v}_2 + b \cdot \mathbf{v}_3 + (c + d) \cdot \mathbf{v}_4$$

Now we can use the hint:

$$\begin{aligned} a + d &= \lambda \\ b - a &= \mu \\ b &= \alpha \\ c + d &= \gamma \end{aligned}$$

Solving this linear system of equations with 4 unknowns and 4 equations, we can get:

$$\begin{aligned} &a = \alpha - \mu \\ &b = \alpha \\ &c = \gamma + \alpha - \lambda - \mu \\ &d = \lambda + \mu - \alpha \end{aligned}$$

And here they are! Whatever vector we want to write as a linear combination of the vectors given in set **c** we should first look at the coefficients of $\mathbf{v}_1 +$

$\mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4$ needed to write that vector as a linear combination of $\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4$ and then use them to calculate the coefficients for the vectors $\mathbf{v}_1 - \mathbf{v}_2, \mathbf{v}_2 + \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_1 + \mathbf{v}_4$.

This proves that we can write all vectors in $\mathbb{R}^4$ as a linear combinations of the vectors in the given set.

Example for c

Here is an example: let's assume we have a vector $\mathbf{v} \in R^4$ that can be written as $\mathbf{v} = 1 \cdot \mathbf{v}_1 + 2 \cdot \mathbf{v}_3 + 3 \cdot \mathbf{v}_3 + 4 \cdot \mathbf{v}_4$. This means we have $\lambda = 1, \mu = 2, \alpha = 3, \gamma = 4$. Now we are trying to write it as a linear combination of the vectors: $\mathbf{v}_1 - \mathbf{v}_2, \mathbf{v}_2 + \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_1 + \mathbf{v}_4$. What coefficients should we use for $a \cdot (\mathbf{v}_1 - \mathbf{v}_2) + b \cdot (\mathbf{v}_2 + \mathbf{v}_3) + c \cdot (\mathbf{v}_4) + d \cdot (\mathbf{v}_1 + \mathbf{v}_4)$? Just fill in the equations from above with the now concrete values for $\lambda, \mu, \alpha, \gamma$:

$$\begin{aligned}
a &= \alpha - \mu & \Rightarrow a &= 3 - 2 = 1 \\
b &= \alpha & \Rightarrow b &= 3 \\
c &= \gamma + \alpha - \lambda - \mu & \Rightarrow c &= 4 + 3 - 1 - 2 = 4 \\
d &= \lambda + \mu - \alpha & \Rightarrow d &= 1 + 2 - 3 = 0
\end{aligned}$$

We have

$$1(\mathbf{v}_1 - \mathbf{v}_2) + 3(\mathbf{v}_2 + \mathbf{v}_3) + 4(\mathbf{v}_4) + 0(\mathbf{v}_1 + \mathbf{v}_4) = 1 \cdot \mathbf{v}_1 + 2 \cdot \mathbf{v}_3 + 3 \cdot \mathbf{v}_3 + 4 \cdot \mathbf{v}_4$$

You can verify this by direct computation.

mkilic

♠