Linear Algebra
In-Class Exercise Week 1

G-07

27 IX 2024

1. Lines in $\mathbb{R}^m$ (in-class) (★★☆)

a) Let $\mathbf{0} \in \mathbb{R}^m$ denote the vector whose entries are all zero. We say that a set L is a line in $\mathbb{R}^m$ if and only if there exists $\mathbf{w} \in \mathbb{R}^m$ with $\mathbf{w} \neq \mathbf{0}$ such that $L = \{\lambda \mathbf{w} : \lambda \in \mathbb{R}\}$. Let now L be a line in $\mathbb{R}^m$ and let $\mathbf{u}$ be an arbitrary non-zero element of L. Prove that $L = \{\lambda \mathbf{u} : \lambda \in \mathbb{R}\}$.

b) For two lines L_1 and L_2 in $\mathbb{R}^m$, prove that we have either $L_1 \cap L_2 = \{\mathbf{0}\}$ or $L_1 \cap L_2 = L_1 = L_2$.

1 Solution

The question defines lines as *sets*. In particular sets of linear combinations of one vector w, which is only scalar multiples of w. This means that in order to prove $L = \{\lambda \cdot u : \lambda \in \mathbb{R}\}$ we have to show the equality of two sets.

How to show the equality of two sets?

In general when we want to show the equality of two sets A, B, we show $A \subseteq B$ **and** $B \subseteq A$. If the two conditions hold together then we can conclude that the two sets are equal.

The following pages have the necessary steps of the proofs and also some explanations to make it clear.

1.1 a)

- We have $u \in L$ and $u \neq 0$ given. Since every element in L can be written as a scalar multiple of w we can write (for some $\lambda_u \in \mathbb{R}$ where $\lambda_u \neq 0$ since $u \neq 0$):
$$u = \lambda_u \cdot w$$
- Now we show that any element in L is also in $\{\lambda \cdot u : \lambda \in \mathbb{R}\}$ *i.e.* $L \subseteq \{\lambda \cdot u : \lambda \in \mathbb{R}\}$
- Let $v \in L$ be arbitrary. Since v is on our line L, by the definition of L we must have some $\lambda_v \in \mathbb{R}$ such that: $v = \lambda_v \cdot w$
- Now write w in terms of u as $w = \frac{1}{\lambda_u} \cdot u$. This gives us $v = \frac{\lambda_v}{\lambda_u} \cdot u$ Since we can write v as a scalar multiple of u this means $v \in \{\lambda \cdot u : \lambda \in \mathbb{R}\}$
- Because v was arbitrary, *i.e.* we were observing any element from the set L, the previous point proves that $L \subseteq \{\lambda \cdot u : \lambda \in \mathbb{R}\}$
- Now we prove the other direction: $\{\lambda \cdot u : \lambda \in \mathbb{R}\} \subseteq L$ For this take an arbitrary vector $v' \in \{\lambda \cdot u : \lambda \in \mathbb{R}\}$. We know $v' = \lambda_{v'} \cdot u$
- We also know that $u = \lambda_u \cdot w$ from the first point.
- Combining the last two points gives us $v' = \lambda_{v'} \cdot u = \lambda_{v'} \lambda_u w$ Since we can write v' as a linear multiple of w, we can conclude that $v' \in L$
- Since v' was arbitrary, *i.e.* we were observing any element from the set $\{\lambda \cdot u : \lambda \in \mathbb{R}\}$, the previous point proves that $\{\lambda \cdot u : \lambda \in \mathbb{R}\} \subseteq L$
- The subset relation in both ways prove that $L = \{\lambda \cdot u : \lambda \in \mathbb{R}\}$

$\square$

1.2 b)

- Let L_1 and L_2 two lines in $\mathbb{R}^m$. This means by definition of a line that there exist $w_1, w_2 \in \mathbb{R}$ such that:

$$L_1 = \{\lambda \cdot w_1 : \lambda \in \mathbb{R}\} \text{ and } L_2 = \{\lambda \cdot w_2 : \lambda \in \mathbb{R}\}$$

- $\mathbf{0} \in L_1 \cap L_2$. $\mathbf{0} \in L_1$ and $\mathbf{0} \in L_2$ because we have $\mathbf{0} = 0 \cdot w_1 = 0 \cdot w_2$
- Now assume $L_1 \cap L_2 \neq \{\mathbf{0}\}$ In words let's assume that there are more than just the vector $\mathbf{0}$ in the intersection of the two lines.
- Since the intersection is not empty by our assumption we have a non-zero vector $u \in L_1 \cap L_2$. This means $u \in L_1$ *and* $u \in L_2$.
- Using the sub-task *a)* we can see that $u \in L_1$ means L_1 can be written as $L_1 = \{\lambda \cdot u : \lambda \in \mathbb{R}\}$ and $u \in L_2$ means L_2 can also be written as $L_2 = \{\lambda \cdot u : \lambda \in \mathbb{R}\}$
- This means, if the two lines share a non-zero element, they are exactly the same lines, *i.e.* $L_1 = L_2$.

$\square$