Linear Algebra
Week 0

G-07

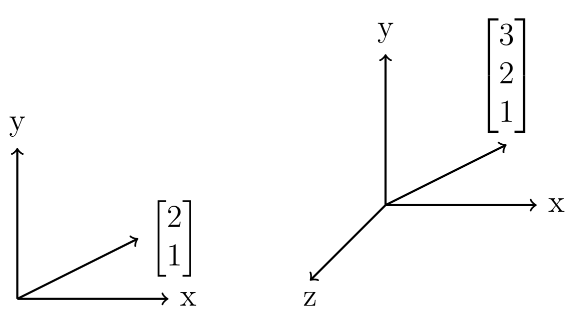
19 IX 2024

1 Vectors

Vectors are one of the main tools we use in linear algebra. For now they are defined as elements of $\mathbb{R}^m, m \in \mathbb{N}$ which represents the m dimensional space.

1.1 Vectors in a Coordinate System

Thinking of vectors as a way to move around in the m dimensional space is a good way of understanding vectors. This might seem easy on a 2 dimensional example. However visualizing vectors and other objects we use in linear algebra like matrices and vector spaces is going to become more difficult as the number of dimensions increases.



(a) in 2 Dimensions

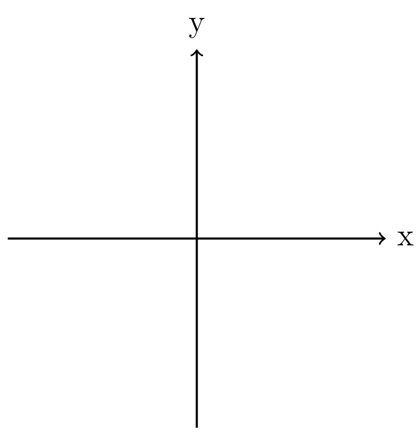
(b) in 3 Dimensions

2 Vector Addition and Scalar Multiplication

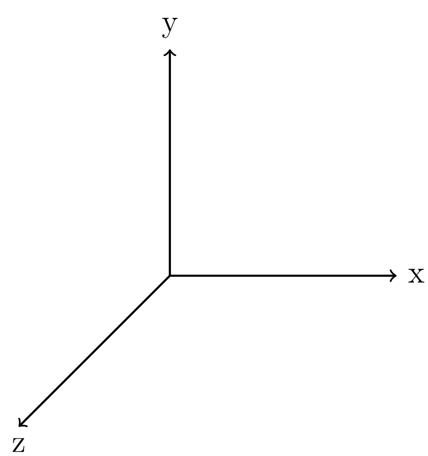
- Addition: $v, u \in \mathbb{R}^m : v + u = \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix} + \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix} = \begin{bmatrix} v_1 + u_1 \\ \vdots \\ v_m + u_m \end{bmatrix} \in \mathbb{R}^m$

- Scalar Multiplication: $\lambda \in \mathbb{R}, v \in \mathbb{R}^m : \lambda \cdot v = \lambda \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} \lambda \cdot v_1 \\ \lambda \cdot v_2 \\ \lambda \cdot v_3 \end{bmatrix} \in \mathbb{R}^m$

Sketch(fill in yourself):



(a)



(b)

You can imagine the vector addition as putting the vectors and to end, in order to reach where you want to go. On the other hand, scalar multiplication is just deciding how many of the same vector you want to use.

3 Linear Combinations

Combine vector addition and scalar multiplication.

Let $\lambda_1, \ldots, \lambda_m \in \mathbb{R}, v_1, \ldots, v_m \in \mathbb{R}^n$ Then:

$$\lambda_1 \cdot v_1 + \cdots + \lambda_m \cdot v_m$$

is a linear combination of the vectors $v_1, \ldots, v_m$.Note that the result is again a vector in the same vector space (*i.e.* the resulting vector has the same dimension). So *we can produce vectors using other vectors!*

A natural question is whether or not we can produce all vectors by using a given set of vectors. Think in 2 dimensions. It might be relatively easy to see that the vectors:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

can be used to produce all 2 dimensional vectors. What about the vectors:

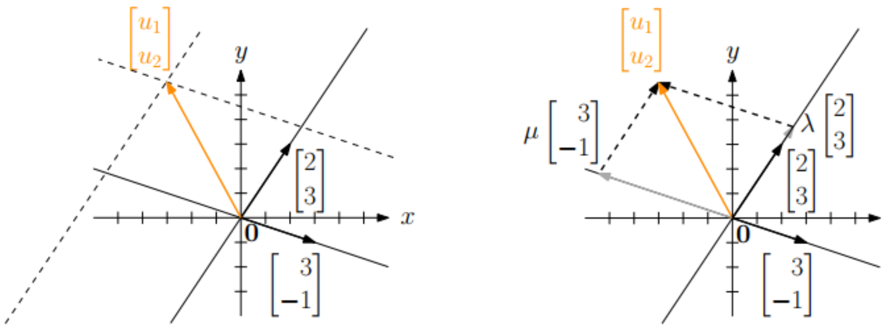
$$\begin{bmatrix} 7 \\ 14 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

Can we produce all 2 dimensional vectors using only these two vectors?

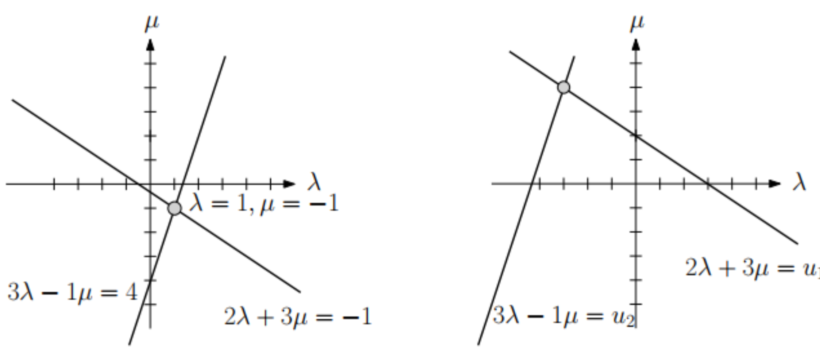
The answer is...

4 Row & Column Pictures

The Column Picture

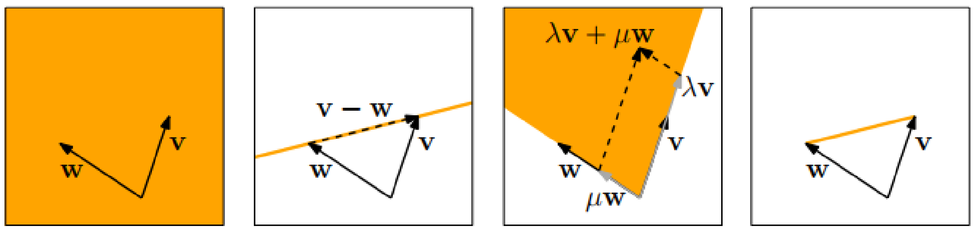


The Row Picture



5 Affine, Conic, and Convex Combinations

- Affine: $\lambda_1 + \cdots + \lambda_m = 1$
- Conic: $\forall i : \lambda_i \geq 0$
- Convex: Affine and Conic



linear (i) affine (ii) conic (iii) convex
combinations of two vectors

Figure 1.10: Two vectors and their combinations

5.1 Affine Combination of Two Vectors

$$\lambda_1 \cdot v_1 + \lambda_2 \cdot v_2 \qquad \text{(since } \lambda_1 + \lambda_2 = 1)$$
$$= \lambda_1 \cdot v_1 + (1 - \lambda_1) \cdot v_2$$
$$= \lambda_1 \cdot v_1 + 1 \cdot v_2 - \lambda_1 \cdot v_2$$
$$= v_2 + \lambda_2(v_1 - v_2)$$

The last line of equation tells us to go to v_2 first and then use any scalar multiple of the vector $v_1 - v_2$ to go somewhere on the line that goes through v_2 and is parallel to $v_1 - v_2$.

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