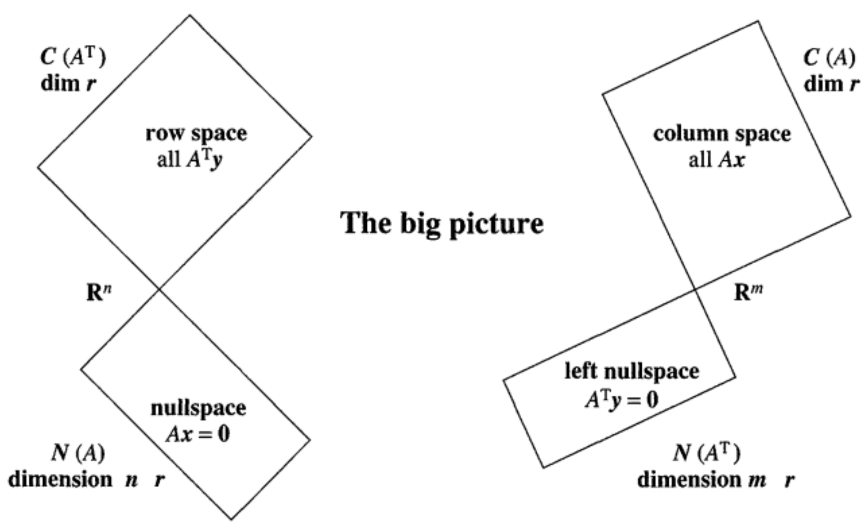
Linear Algebra
Week 7

G-07

7 XI 2024

1 Fundamental Subspaces

Matrices are the backbones of linear algebra and more, whether it's multivariate calculus, or machine learning, we can represent various objects as matrices. The 4 fundamental subspaces associated with a matrix will give us important insights about the matrix at hand. They are: Column Space $\mathbf{C}(A)$, Row Space $\mathbf{R}(A)$, Nullspace $\mathbf{N}(A)$ and the Left Nullspace $\mathbf{LN}(A)$. Here is an overview from Professor Strang's book Introduction to Linear Algebra. You can find him explain this on YouTube, I totally recommend watching that particular part of his lecture. (The dimensions below should be $n - r$ and $m - r$ for $\mathbf{N}$(A) and $\mathbf{N}(A^\top)$)



Gilbert St. (2009), Introduction to Linear Algebra, 4th Edition, Page 187

1.1 Column Space $\mathbf{C}(A)$

The column space of a matrix $A \in \mathbb{R}^{m \times n}$ is defined as follows:

$$\mathbf{C}(A) = \{A\mathbf{x} \mid \mathbf{x} \in \mathbb{R}^n\} \subseteq \mathbb{R}^m$$

This means, the column space of a matrix is the set of all linear combinations of its columns. The column space is also referred as the *Image* of A ($\mathbf{Im}(A)$). This makes more sense if you think of A as a linear transformation.

Theorem 4.25 (Informal) The independent columns of A, which you can detect using REF and Gauss Jordan, form a basis for the column space of A. Therefore we have

$$dim(\mathbf{C}(A)) = r = \mathbf{rank}(A)$$

Intuition: The column space is equal to the span of A's columns, this is the name for *all linear combinations*. We know that taking linearly dependent vectors out of a set does not change the span of the set. Hence the linearly independent columns indeed form a basis of $\mathbf{C}$(A). The rank is defined as the number of linearly independent columns so $\mathbf{C}$(A) has dimension $\mathbf{rank}(A)$.

1.2 Row Space $\mathbf{R}(A)$

The row space of a matrix $A \in \mathbb{R}^{m \times n}$ is defined as

$$\mathbf{R}(A) = \{\mathbf{y}^\top A \mid \mathbf{y} \in \mathbb{R}^m\} \subseteq \mathbb{R}^n$$

So we define it as the linear combinations of all rows. A more natural approach would be to define it as

$$\mathbf{R}(A) = \mathbf{C}(A^\top) \subseteq \mathbb{R}^n$$

A key takeaway is that $\mathbf{R}(A)$ is a subspace of $\mathbb{R}^n$.

Nice, so how do we calculate the row space, *i.e.* find a basis of it? We could find a basis of $\mathbf{C}(A^\top)$. This would work and we do use this but we can find a basis using the REF of A as well. Let's see how to do it in a way that sparks joy. We need a lemma first.

Lemma 4.27 Let A be an $m \times n$ matrix and M an invertible $m \times m$ matrix. Then $\mathbf{R}(A) = \mathbf{R}(MA)$.

Intuition: The row space of a matrix does not change if we multiply it with an invertible matrix *on the left*.

Proof: We have

$$\mathbf{R}(A) = \mathbf{C}(A^\top) \stackrel{!}{=} \mathbf{C}(A^\top M^\top) = \mathbf{C}((MA)^\top) = \mathbf{R}(MA)$$

We know the first and last equations hold (see above). The third one is the property of the matrix transpose. What we need to show is the second one. So we set $B := A^\top$ and show $\mathbf{C}(B) = \mathbf{C}(BN)$, from the lecture notes:

$$\begin{array}{c} \mathbf{v} \in \mathbf{C}(B) \\ \Updownarrow \\ \mathbf{v} = B\mathbf{x} \text{ for some } \mathbf{x} \in \mathbb{R}^m \\ \Uparrow \qquad\qquad \Downarrow \\ \mathbf{v} = BN\mathbf{y} \text{ for some } \mathbf{y} \in \mathbb{R}^m \\ \Updownarrow \\ \mathbf{v} \in \mathbf{C}(BN). \end{array} \quad \leftarrow \mathbf{y} := N^{-1}\mathbf{x} \quad \Leftrightarrow \quad \mathbf{x} := N\mathbf{y}$$

□

Theorem 4.28 (Informal) Let R_0 be in REF, be the result of the Gauss Jordan elimination on A. Then the nonzero rows of R_0 are a basis of $\mathbf{R}$(A). Therefore we have:

$$dim(\mathbf{R}(A)) = r = \mathbf{rank}(A)$$

Intuition: We represent the Gauss-Jordan elimination with an invertible matrix M as $R_0 = MA$. (Theorem 3.21). Above we prove that multiplying A on the left with an invertible matrix does not change the row space, hence $\mathbf{R}(A) = \mathbf{R}(MA) = \mathbf{R}(R_0)$. This means a basis for the latter is also a basis for the former. We can easily read off a basis of R_0 which is the nonzero rows since these rows are linearly independent (private nonzeros) and the 0-rows do not contribute to the span. We know there are $r = \mathbf{rank}(A)$ downward steps in REF so we have $dim(\mathbf{R}(R_0)) = dim(\mathbf{R}(A)) = r$

Theorem 4.29 let A be an $m \times n$ matrix. Then

$$\mathbf{rank}(A) = \mathbf{rank}(A^\top)$$

Intuition: The row rank is equal to the column rank! You can deduce that from $dim(\mathbf{C}(A)) = dim(\mathbf{R}(A)) = dim(\mathbf{C}(A^\top))$. The first equation holds because both sides are equal to **rank(A)** $= r$ as we show above. The third equation holds because $\mathbf{R}(A) = \mathbf{C}(A^\top)$ (again, see above). So $A^\top$ has so many independent columns as A.

There is a nice link to the CR-Decomposition:

Corollary 4.30. Let $A = CR$ be the CR decomposition of A (Section 2.2.3). By Theorem 4.25, the columns of C form a basis of the column space **C**(A). By Theorem 4.28 together with Theorem 3.24, the rows of R form a basis of the row space **R**(A). Both spaces have the same dimension $r = \mathbf{rank}(A) = \mathbf{rank}(A^\top)$.

1.3 Nullspace $\mathbf{N}(A)$

Definition 4.31 Let A be an $m \times n$ matrix. The nullspace of A is the set of all solutions of $A\mathbf{x} = \mathbf{0}$:

$$\mathbf{N}(A) = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\} \subseteq \mathbb{R}^n$$

This means we have all vectors that are taken to **0** by the linear transformation represented by A. This is why the nullspace is also referred as the kernel of A: **Ker**(A).

Lemma 4.32 Let A be a $m \times n$ matrix. Then **N**(A) is a subspace of $\mathbb{R}^n$. (Proof in lecture notes, you can also do this yourself.)

A natural question is, how do we calculate a basis. At this point the Gauss Jordan elimination helps us again. Yet, similar to what we did for the row space, we have to show a lemma first.

Lemma 4.33. *Let A be an $m \times n$ matrix and M an invertible $m \times m$ matrix. Then A and MA have the same nullspace $\mathbf{N}(A) = \mathbf{N}(MA)$.*

Proof. For $\mathbf{x} \in \mathbb{R}^n$, we have

$$\begin{array}{ccccccc} \mathbf{x} \in \mathbf{N}(A) & \Leftrightarrow & A\mathbf{x} = \mathbf{0} & \Rightarrow & MA\mathbf{x} = M\mathbf{0} & & \\ & & \Uparrow & & \Downarrow & & \\ & & M^{-1}MA = M^{-1}\mathbf{0} & \Leftarrow & MA\mathbf{x} = \mathbf{0} & \Leftrightarrow & \mathbf{x} \in \mathbf{N}(MA). \end{array}$$

□

Now we are ready to find a basis of the nullspace. Here I am going to take the example from the lecture notes and provide an explanation in other words. We have:

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 4 & 1 & 4 \\ 3 & 6 & 2 & 5 \end{bmatrix} \to R_0 = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thanks to the lemma above we can see that $\mathbf{N}(A) = \mathbf{N}(MA) = \mathbf{N}(R_0)$ Also we know that $R_0\mathbf{x} = 0$ hold for the 0 rows regardless of $\mathbf{x}$ because then we have an equation that tells us $\mathbf{0}^\top \mathbf{x} = 0$.

$$\mathbf{N}\left(\begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 4 & 1 & 4 \\ 3 & 6 & 2 & 5 \end{bmatrix}\right) = \mathbf{N}\left(\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right) = \mathbf{N}\left(\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix}\right).$$

The nullspace of A, or equivalently the nullspace of R, consists of vectors $\mathbf{x}$ that satisfy $R\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

First of all realize that this equation $R\mathbf{x} = \mathbf{0}$ represents a system of linear equations where we have 4 unknowns and 2 equations. We also know that the rank is equal to 2. The system is underdetermined and we have 2 free variables. That is, we can choose two variables as we wish and calculate the other two such that all variables together satisfy the equation $R\mathbf{x} = \mathbf{0}$.

As we know the matrix vector multiplication is just the linear combination of the columns of the matrix. If we separate the columns of A in a wise way so that we have the identity matrix I and another matrix Q, our equation looks like

$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_3\end{bmatrix}+\begin{bmatrix}2 & 3\\0 & -2\end{bmatrix}\begin{bmatrix}x_2\\x_4\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}.$$

Basic arithmetic tells us:

$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_3\end{bmatrix}=-\begin{bmatrix}2 & 3\\0 & -2\end{bmatrix}\begin{bmatrix}x_2\\x_4\end{bmatrix}$$

If you do the vector multiplication the equation boils down to the following linear system of equations:

$$\begin{aligned}x_1 &= -2x_2 - 3x_4\\ x_3 &= 2x_4\end{aligned}$$

Remember that we have 2 free variables *i.e.* we are free to choose 2 variables as we wish. You can choose the values of any two variables and calculate the other two as long as you obey the dependencies. (*e.g.* you can't choose $x_3 = 0$ and $x_4 = 1$ this would be a contradiction) However here choosing x_2 and x_4 as our free variables is the easiest choice, since then plugging these values into the equations above, we directly obtain the values of x_1 and x_3 as well. In the lecture you have seen a table as follows:

		special solutions	
free variables	$\underbrace{\begin{bmatrix}x_2\\x_4\end{bmatrix}}_{\mathbf{x}(Q)}$	$\underbrace{\begin{bmatrix}1\\0\end{bmatrix}}_{\mathbf{v}_1(Q)}$	$\underbrace{\begin{bmatrix}0\\1\end{bmatrix}}_{\mathbf{v}_2(Q)}$
basic variables	$\underbrace{\begin{bmatrix}-2 & -3\\0 & 2\end{bmatrix}}_{-Q}\underbrace{\begin{bmatrix}x_2\\x_4\end{bmatrix}}_{\mathbf{x}(Q)}=\underbrace{\begin{bmatrix}x_1\\x_3\end{bmatrix}}_{\mathbf{x}(I)}$	$\underbrace{\begin{bmatrix}-2\\0\end{bmatrix}}_{\mathbf{v}_1(I)}$	$\underbrace{\begin{bmatrix}-3\\2\end{bmatrix}}_{\mathbf{v}_2(I)}$
nullspace equation	$\mathbf{0}=\underbrace{\begin{bmatrix}1 & 2 & 0 & 3\\0 & 0 & 1 & -2\end{bmatrix}}_{R}\underbrace{\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}}_{\mathbf{x}}$	$\underbrace{\begin{bmatrix}-2\\1\\0\\0\end{bmatrix}}_{\mathbf{v}_1}$	$\underbrace{\begin{bmatrix}-3\\0\\2\\1\end{bmatrix}}_{\mathbf{v}_2}$

Here the vector consisting of free variables is noted as $\mathbf{x}(Q)$. The dependent variables are named *basic variables* and the vector consisting of the basic variables is noted by $\mathbf{x}(I)$.

This table from the lecture notes summarizes the whole approach and offers that we choose the free variables one time as $x_2 = 1, x_4 = 0$ and the second time as $x_2 = 0, x_4 = 1$. These choices provide us with two solutions for $R\mathbf{x} = \mathbf{0}$ as well as $A\mathbf{x} = \mathbf{0}$, ($\mathbf{v}_1$ and $\mathbf{v}_2$ from the table) which are a basis of the nullspace $\mathbf{N}(A)$. In the literature these solutions are referred as the ***particular solutions***. The basis is:

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

You can read **Lemma 4.34** for a formalization of this approach and a formal proof, why the set we claim is a basis, is actually a basis. As you read it, don't forget that $\mathbf{x}$(Q) and $\mathbf{x}$(I) are just smaller vectors that consist of some entries of the vector $\mathbf{x}$.

Theorem 4.35 (Informal) Let R_0 be in REF, the result of Gauss-Jordan elimination on $A \in \mathbb{R}^{m\times n}$. Let R be the matrix R_0 in RREF, in reduced form. The vectors, $\mathbf{v}_1, \ldots, \mathbf{v}_{n-r}$ constructed as above form a basis of $\mathbf{N}(A) = \mathbf{N}(R_0) = \mathbf{N}(R)$ Hence:

$$dim(\mathbf{N}(A)) = n - r = n - \mathbf{rank}(A)$$

Intuition: We have so many rows in R in RREF as we have downward steps in R_0, which is equal to the rank of A. Therefore in the equation $R\mathbf{x} = \mathbf{0}$, the matrix R has dimensions $r \times n$. This means we have $n - r$ free variables. We can choose $\mathbf{x}(Q)$ to be one of the $n - r$ unit vectors each time to get a particular solution. Therefore we have $dim(\mathbf{N}(A)) = n - r = n - \mathbf{rank}(A)$.

1.4 Left Nullspace $\mathbf{LN}(A)$

The left nullspace of a matrix is the set of *row vectors*:

$$\mathbf{LN}(A) = \{\mathbf{y} \in \mathbb{R}^m \mid \mathbf{y}A = \mathbf{0}^\top\} \subseteq \mathbb{R}^m$$

but if you take the transpose of both sides of the equation and write $\mathbf{x} := y^\top$ where $\mathbf{x}$ is then a column vector, you can see that the above set is equivalent to

$$\mathbf{LN}(A) = \{\mathbf{x} \in \mathbb{R}^m \mid A^\top\mathbf{x} = \mathbf{0}\} \subseteq \mathbb{R}^m$$

Sounds familiar? This is the nullspace of $A^\top$. Therefore a natural way to define the left nullspace of a matrix is

$$\mathbf{LN}(A) := \mathbf{N}(A^\top) \subseteq \mathbb{R}^m$$

With this definition you can immediately realize that $\mathbf{NL}(A)$ is a subspace of $\mathbb{R}^m$. (**Lemma 4.37**).

Theorem 4.38 (Informal) Let A be an $m \times n$ matrix with $\mathbf{rank}(A) = r$ and let $R_0 = MA$ be in REF, the result of Gauss Jordan elimination on A. The last $m - r$ rows of the matrix M are a basis of $\mathbf{LN}(A)$ and therefore the left nullspace of A has dimension:

$$dim(\mathbf{LN}(A)) = m - r = m - \mathbf{rank}(A)$$

Intuition: We know that the last $m - r$ rows of R_0 are 0 rows since R_0 is in REF and there are r nonzero rows (r = #downwards steps = $\mathbf{rank}$(A)). $R_0 = MA$ tells us that the last $m - r$ rows M, when multiplied with the matrix A, give us the 0 rows in the bottom of R_0. Let $\mathbf{u}_1, \ldots \mathbf{u}_m$ be the rows of M:

$$R_0 = MA = \begin{bmatrix} \text{———}\mathbf{u}_1^\top A\text{———} \\ \text{———}\mathbf{u}_2^\top A\text{———} \\ \vdots \\ \text{———}\mathbf{u}_r^\top A\text{———} \\ \text{———}\mathbf{u}_{r+1}^\top A\text{———} \\ \vdots \\ \text{———}\mathbf{u}_m^\top A\text{———} \end{bmatrix}$$

If the last $m-r$ rows are zero this means $\mathbf{u}_{r+1}, \ldots \mathbf{u}_m \in \mathbf{LN}(A)$. Now we need to show that this is a basis. The vectors are linearly independent because M is invertible. (Inverse Theorem tells us all columns of an invertible matrix are independent. We learned that row rank equals to column rank. Therefore all rows of M are linearly independent.) We also know that $dim(\mathbf{LN}(A)) = dim(\mathbf{N}(A^\top)) = m - \mathbf{rank}(A^\top) = m - \mathbf{rank}(A) = m - r$. We have $m-r$ linearly independent vectors in $m-r$ dimensional space. So these vectors must be a basis by *Lemma 4.24*.

1.5 The solution Space of $A\mathbf{x} = \mathbf{b}$

Definition 4.39 (Solution space). *Let A be an $m \times n$ matrix and $\mathbf{b} \in \mathbb{R}^m$. The set*

$$\mathbf{Sol}(A, \mathbf{b}) := \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{b}\} \subseteq \mathbb{R}^n$$

is the solution space *of $A\mathbf{x} = \mathbf{b}$.*

You should be careful, if $\mathbf{b} \neq \mathbf{0}$ then $\mathbf{Sol}(A, \mathbf{b})$ is not a subspace because it does not include the zero vector.

The next theorem tells us the important relation between the nullspace and the solution space.

Theorem 4.40. *Let A be an $m \times n$ matrix, $\mathbf{b} \in \mathbb{R}^m$. Let $\mathbf{s}$ be some solution of $A\mathbf{x} = \mathbf{b}$. Then*

$$\mathbf{Sol}(A, \mathbf{b}) = \{\mathbf{s} + \mathbf{x} : \mathbf{x} \in \mathbf{N}(A)\}.$$

Intuition: If you shift the nullspace from the origin by a solution, then you have the space of all solutions.

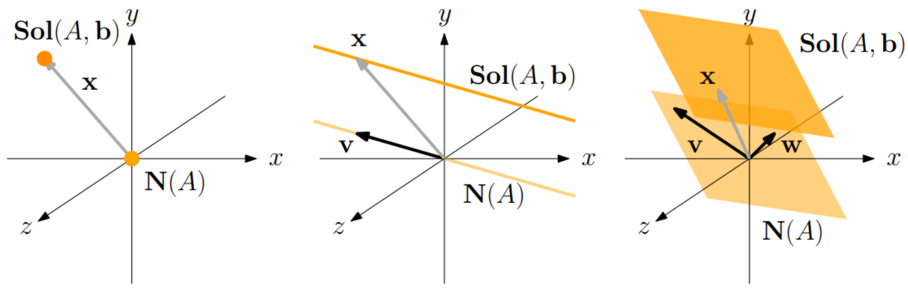
Proof. We first show that every solution $\mathbf{y} \in \mathbf{Sol}(A, \mathbf{b})$ is of the form $\mathbf{s} + \mathbf{x}$ with $\mathbf{x} \in \mathbf{N}(A)$. Indeed, we can write $\mathbf{y}$ as

$$\mathbf{y} = \mathbf{s} + \underbrace{(\mathbf{y} - \mathbf{s})}_{\mathbf{x}},$$

and due to $A\mathbf{x} = A\mathbf{y} - A\mathbf{s} = \mathbf{b} - \mathbf{b} = \mathbf{0}$, we have $\mathbf{x} \in \mathbf{N}(A)$. For the other direction, we show that every vector $\mathbf{y}$ of the form $\mathbf{y} = \mathbf{s} + \mathbf{x}$ with $\mathbf{x} \in \mathbf{N}(A)$ is in $\mathbf{Sol}(A, \mathbf{b})$. For this, we compute

$$A\mathbf{y} = A\mathbf{s} + A\mathbf{x} = \mathbf{b} + \mathbf{0} = \mathbf{b}.$$

$\square$



Number of Solutions: In the lecture notes there is a nice discussion about the number of solutions to a system of equations $A\mathbf{x} = \mathbf{b}$. The terms "overdetermined" and "underdetermined" are also defined there. You should read that part to have a general understanding of the linear systems of equations. In general there can be 0, 1, or infinitely many solutions and it is determined mainly by the dimensions and the rank of the matrix.

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Second part of the lecture starts here. The references to the lecture notes are going to be to the second part of the lecture notes.

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2 Orthogonality

This section is about orthogonality of vectors and vector spaces. The first definition defines what it means to be orthogonal for two vector spaces:

Definition 5.1.1. Two vectors $v, w \in \mathbb{R}^n$ are called orthogonal if $\mathbf{v}^\top \mathbf{w} = \sum_{i=1}^{n} v_i w_i = 0$. Two subspaces V and W are orthogonal if for all $\mathbf{v} \in V$ and $\mathbf{w} \in W$, the vectors v and w are orthogonal.

Intuition: In other words we name two vetor spaces orthogonal if all vectors in one vector space is orthogonal to all vectors in the other vector space.

We can make 3 important statements about the orthogonal vector spaces, here without the proofs:

1. *Bases are enough to determine if two vector spaces are orthogonal:*

   Let $\mathbf{v}_1, \ldots, \mathbf{v}_k$ be the basis of subspace V and let $\mathbf{w}_1, \ldots, \mathbf{w}_k$ be the basis of subspace W. Then V and W are orthogonal *if and only if* $\mathbf{v}_i$ and $\mathbf{w}_j$ are orthogonal for all $i \in [k]$ and $j \in [l]$. **(Lemma 5.1.2.)**

2. *Bases of orthogonal spaces are linearly independent:*

   Let V and W be two orthogonal subspaces of $\mathbb{R}^n$. Let $v_1, \ldots, v_k$ be a basis of subspace V. Let $w_1, \ldots, w_l$ be a basis of subspace W. The set of vectors $\{v_1, \ldots, v_k, w_1, \ldots, w_l\}$ are linearly independent. (**Lemma 5.1.3.**)

3. *Combining orthogonal subspaces give us another subspace, the dimension is the sum of the dimension of the orthogonal subspaces:*

   Let V and W be orthogonal subspaces. Then $V \cap W = \{\mathbf{0}\}$. Moreover, if $dim(V) = k$ and $dim(W) = l$, then

   $$dim(\{\lambda v + \mu w \mid \lambda, \mu \in \mathbb{R}, v \in V, w \in W\}) = k + l \leq n$$

   (**Corollary 5.1.4.**)

2.1 Orthogonal Complement

Now we are ready to define an important concept:

Definition 5.1.5. Let V be a subspace of $\mathbb{R}^n$. We define the orthogonal complement of V as:

$$V^{\perp} = \{\mathbf{w} \in \mathbb{R}^n \mid \mathbf{w}^{\top}\mathbf{v} = 0 \text{ for all } \mathbf{v} \in V\}$$

In words, the orthogonal complement of V includes all vectors that are orthogonal to all vectors in V. All subspaces orthogonal to V are in $V^{\perp}$.

The following theorem tells us that a vector space and its orthogonal complement, complete each other to the whole vector space.

Theorem 5.1.7. *Let V, W be orthogonal subspaces of* R^n.

The following statements are equivalent.

(i) $W = V^{\perp}$.
(ii) $\dim(V) + \dim(W) = n$.
(iii) *Every* $u \in \mathrm{R}^n$ *can be written as* $u = v + w$ *with unique vectors* $v \in V$, $w \in W$.

An important takeaway is for $V \subseteq \mathbb{R}^n$:

$$\mathbb{R}^n = V + V^{\perp} = \{\mathbf{v} + \mathbf{w} \mid \mathbf{v} \in V, \mathbf{w} \in V^{\perp}\}.$$

We also have:

Lemma 5.1.8. Let V be a subspace of $\mathbb{R}^n$. Then $V = (V^{\perp})^{\perp}$.

This seems trivial but actually needs a proof. You know where to find it.

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This discussion brings us back to the "Big Picture of Linear Algebra" from page 1. The 4 fundamental subspaces of A are linked together:

$$\mathbf{N}(A) = \mathbf{C}(A^{\top})^{\perp} \text{ and } \mathbf{N}(A^{\top}) = \mathbf{C}(A)^{\perp}$$

In words, the nullspace of A and the row space of A as well as the left nullspace of A and the column space of A are respectively orthogonal complements. The former is proven in **Theorem 5.1.6.**

There is one last Lemma, which is important and relevant for a few upcoming topics *e.g.* for least squares.

Lemma 5.1.11. *Let $A \in \mathrm{R}^{m \times n}$. Then $N(A) = N(A^T A)$ and $C(A^T) = C(A^T A)$.*

Proof. If $x \in \mathrm{N}(A)$ then $Ax = 0$ and so $A^\top Ax = 0$, thus $x \in \mathrm{N}(A^\top A)$. The other implication is more interesting.

If $x \in \mathrm{N}(A^\top A)$ then $A^\top Ax = 0$. This implies that $x^\top A^\top Ax = x^\top 0 = 0$. But $x^\top A^\top Ax = (Ax)^\top (Ax) = \|Ax\|^2$ so Ax must be a vector with norm 0 which implies that $Ax = 0$ and so $x \in \mathrm{N}(A)$.

For the second statement we utilize Corollary 5.1.9. We have

$$C(A^T) = N(A)^\perp = N(A^T A)^\perp = C((A^T A)^T) = C(A^T A).$$

The proof uses a very elegant method. If you show that $\|Ax\|_2 = 0$ or $\|Ax\|_2^2 = 0$, which are actually equivalent, then this implies $Ax = 0$. Because only the vector $\mathbf{0}$ has norm 0. So if you have an equation in form of $A^\top Ax = 0$ or $x^\top A^\top A = 0$ you can multiply both sides with x or $x^\top$ respectively from left/right to acquire $x^\top A^\top Ax = \|Ax\|_2^2 = 0$.

3 Hints

1. Solved In Class You can also see the section on nullspace above.

2. Use your findings from the previous subtasks if you are stuck.

3. No hints.

4. Define $flatten(A)$ as the vector with m^2 elements, that includes all elements of A in one column. (but how?) Then $Trace(A) = 0$ if and only if $flatten(A) \cdot flatten(I) = 0$. Does this sound familiar?

5. Think of the three options: How can 3 points exist in space? They can be on top of each other, *i.e.* they can be identical. They can lie on the same line, *i.e.* they can be collinear, or they can be the corners of a triangle. Now show this intuition mathematically.

6. No hints.

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