Linear Algebra
Week 4

G-17

16 X 2025

1 CR Decomposition

A matrix decomposition is breaking a matrix in several matrices, which usually each give us information about the decomposed matrix. CR decomposition is the first of many matrix decompositions you are going to encounter during the course.

Theorem 2.46 (CR decomposition) *Let A be an $m \times n$ matrix of rank r. Let C be the $m \times r$ submatrix of A containing the independent columns. Then there exists a unique $r \times n$ matrix R' such that:*

$$A = CR'$$

Let's see why this is true.

- The column space of A: $\mathbf{C}(\mathrm{A}) = \mathbf{C}(\mathrm{C})$: Column space of C. This is because we have all linearly independent columns of A in C.
- The first point means that we can write the columns of A as a linear combination of the columns of C. This is because the columns of A themselves are also in the column space of A!
- With this point of view, in R' at i-th column, we store how much of which column of C we need to generate the i-th column of A.

- We know by Lemma 1.24 that there is only one way of writing a vector as a linear combination of linearly independent vectors. Therefore R' is unique.

$$\underbrace{\begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 4 & 1 & 4 \\ 3 & 6 & 2 & 5 \end{bmatrix}}_{A,\ 3\times 4} = \underbrace{\begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix}}_{C,\ 3\times 2} \underbrace{\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix}}_{R',\ 2\times 4}.$$

The 4^{th} column of R tells us that we need 3 times the 1^{st} column and -2 times the 2^{nd} column of C, in order to generate the 4^{th} column of A.

We know that A and C have the same column space. We will see that A and R' have the same row space later. However to convince yourself that $\mathbf{R}(A) = \mathbf{R}(R')$ now, you can view the matrix multiplication CR' as *the linear combination of the rows of* R'. We have

$$(\text{1st row of A}) = 1 \ \cdot (\text{1st row of } R') + 0 \cdot (\text{2nd row of } R')$$

where the coefficients 1 and 0 are the first row of C. You use the rows of R' to generate the rows of the matrix A so intuitively we have $\mathbf{R}(A) = \mathbf{R}(R')$.

Keep in mind: You can also use the CR decomposition to compress the matrices. A has $m \cdot n$ entries whereas you only have to store $(m+n) \cdot r$ entries in the CR-decomposition.

2 Inverse Matrices

Until now we have seen how to add matrices, how to multiply them, but not how to take back the multiplication by a matrix. If you see a matrix as a linear transformation, this taking back operation would correspond to undoing the effect of the linear transformation. Did your matrix stretch the space and doubled every distance in x_1 direction? Then the inverse would shrink the space in the same direction so that everything gets back to "normal", or in other words everything gets back to the way as they were before you applied any transformations.

2.1 Functions and Bijectivity

Think of $f : X \to Y$. For us to be able to undo the operation, we have to have every possible input to be mapped to exactly one possible output. As you know from other lectures, we call such functions ***bijective*** since they are both injective and surjective.

Now we can convince us that a matrix has to be square in order to be undoable. This means in particular that the matrix transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$ cannot be bijective if $m \neq n$. Let's see this for both cases $m < n$ (short and wide matrix) and $n < m$ (skinny and tall matrix).

To see this we can look at an example. You can find the formal proofs (***Lemma 2.50*** and ***Lemma 2.51***) in the lecture notes.

Case ($m < n$): If we have $m < n$ we have a short and wide matrix with the matrix transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$. We are trying to squeeze everything in some higher dimension to some lower dimension. Naturally some things must overlap. So if we try to revert this squeezing we can't know what goes where. Here is an example with $n = 3$ and $m = 2$. We are doing projection and transforming $\mathbb{R}^3 \to \mathbb{R}^2$

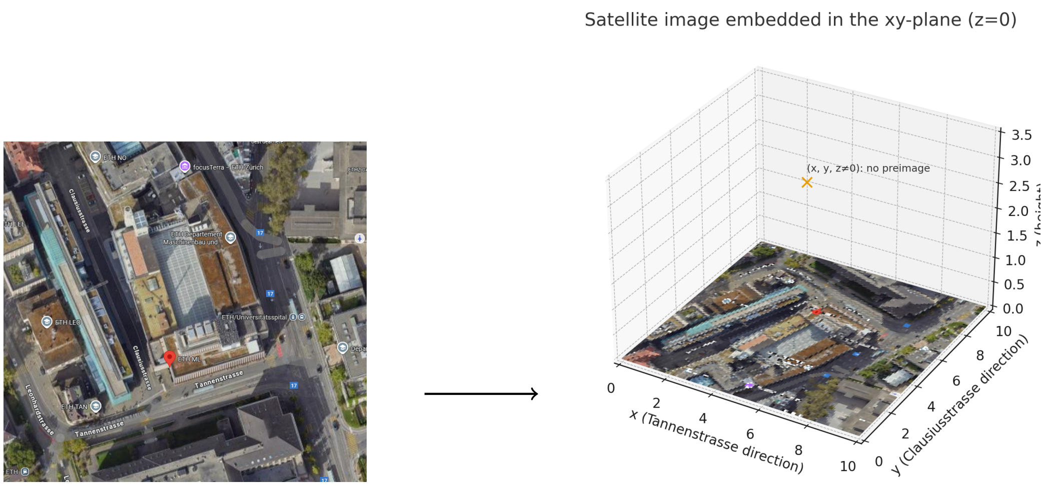
3D Real World View

2D Squeezed Plane

Figure 1: Mapping from a 3D space (left) to its projection in $\mathbb{R}^2$ plane (right).

Now if you are asked to take this operation back just by looking at the 2D image, you wouldn't be able to do so. For example you can not know how many stories the building has or what color the side facades have. This operation is not injective and hence not bijective (not undoable).

***Case* $(n < m)$**: If we have $n < m$ we have a skinny and tall matrix with the matrix transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$. We are trying to fill out a larger space of m dimensions using inputs from a smaller dimension n but we simply do not have enough input to fill the whole space. This will lead to some parts in the larger output space to have no input mapped to them. Therefore our operation will not be surjective and hence not bijective (not undoable). Here is an example with $n = 2$ and $m = 3$.



2D Squeezed Plane

2D image embedded in 3D

Figure 2: Mapping from a 2D plane (left) to its embedding in $\mathbb{R}^3$ (right).

Now if you were asked to take this operation back for every point in $\mathbb{R}^3$ you wouldn't be able to do so. This transformation is essentially

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

and it is irreversible if you have $z \neq 0$ in the 3D space. As you can see only the xy-plane can be mapped back but the marked point outside of the plane can not be mapped back to the input domain. This operation is not surjective and hence it is not undoable (not bijective).

Keep in mind that these examples are here for intuition. See the formal reasoning in the lecture notes.

2.2 Inverse of a Bijective Linear Transformation

Now we know that the only candidates for invertible linear transformations are the linear transformations $T : \mathbb{R}^m \to \mathbb{R}^m$.

Lemma 2.52 (The inverse of a bijective linear transformation). Let $T : \mathbb{R}^m \to \mathbb{R}^m$ be a bijective linear transformation. Then its inverse $T^{-1} : \mathbb{R}^m \to \mathbb{R}^m$ is also a linear transformation (and bijective by Fact 2.49).

Now we know that the inverse of a linear transformation is also a linear transformation. This means the inverse of a linear transformation can also be represented with a matrix. This justifies the existence of inverse matrices. You can see the short proof of linearity in the lecture notes.

Now we formally state the equivalence we have seen in the previous subsection:

Lemma 2.53 (Bijective matrix transformations). Let A be an $m \times m$ matrix. The following three statements are equivalent.

1. $T_A : \mathbb{R}^m \to \mathbb{R}^m$ is bijective.
2. There is an $m \times m$ matrix B such that $BA = I$.
3. The columns of A are linearly independent.

Most importantly we know that the right inverse of an $m \times m$ matrix A is also its left inverse.

Lemma 2.54. Let A, B be $m \times m$ matrices such that $BA = I$. Then also $AB = I$.

2.3 The Inverse of a Matrix

Let's formally define an invertible matrix:

Definition 2.55 (Invertible and singular matrix). Let A be an $m \times m$ matrix. A is called *invertible* if there is an $m \times m$ matrix B such that $BA = I$. (By Lemma 2.53 this is the case if and only if A has linearly independent columns.) Otherwise, A is called *singular*.

We have further alternative definitions of the invertible matrix:

Observation 2.56 (Alternative definitions of an invertible matrix). Let A be an $m \times m$ matrix. A is invertible if and only if there is an $m \times m$ matrix B satisfying one (and therefore all) of the following equivalent conditions.

1. $AB = I$.
2. $BA = I$. (This means that A is invertible according to Definition 2.55.)
3. $AB = BA = I$.

Moreover, the matrix B is unique.

Definition 2.57 (Inverse matrix). Let A be an $m\times m$ matrix. A is invertible if and only if there exists an $m \times m$ matrix B such that $BA = I$ (or $AB = I$, or $AB = BA = I$). In this case, the matrix B is unique and called the *inverse* of A. We denote it by A^{-1}.

As a consequence, A^{-1} is also invertible, and its inverse is the original matrix A.

How to calculate the inverse?

Case 1×1**.**

$$M = \begin{bmatrix} a \end{bmatrix} \quad \Rightarrow \quad M^{-1} = \begin{bmatrix} \frac{1}{a} \end{bmatrix} \quad \text{(if } a \neq 0\text{)}.$$

Hence, the inverse exists unless $a = 0$.

Case 2×2**.**

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \Rightarrow \quad M^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad \text{(if } ad - bc \neq 0\text{)}.$$

The inverse for 2×2 matrices exist, if and only if when $ad - bc \neq 0$. This is equivalent to say that the inverse of the 2×2 matrix exists, if and only if the columns of it are linearly independent. (Think about this!).

This formula might feel like it came out of nowhere, but these two examples above are only special cases of a general rule to calculate the inverse of a matrix. The formula utilizes some linear algebra tools we have not acquired yet, like determinants and cofactors, but don't worry, we are going to get there.
Last but not least we list a few important porperties of the inverse:

Lemmas 2.58–2.60. In the following, let A and B be $m \times m$ invertible matrices.

1. $(A^{-1})^{-1} = A$ (Lemma 2.58)
2. $(AB)^{-1} = B^{-1}A^{-1}$ (Lemma 2.59)
3. $(A^{\top})^{-1} = (A^{-1})^{\top}$ (Lemma 2.60)

You can find proofs of this facts in the lecture notes but I recommend to think about the proofs before you read them.

3 Hints

1. Solved In Class

2. Solved In Class

3. **a)** Use associativity **b)** be careful here A is not necessarily invertible **c)** no hints

4. No hints.

5. **a)** use induction over k **b)** Go to the assignment from week 2 exercise 2c **c)** This is a one liner think clever **d)** you have already seen a type of matrix in the course, which can be a self inverse, find it. (you are of course free to come up with another one) **e)** If you can modify the one from the previous subtask, go for it.

6. (Challenging) **a)** use formula **b)** You have to prove two directions. Drawing a sketch or writing down such a matrix might help. For the proof of one direction, the proof for Theorem 3.5. $\neg(i) \Rightarrow \neg(ii)$ might inspire you. **c)** Assume for a contradiction that the inverse of a lower triangular matrix is not lower triangular. What is the issue here? Argue formally for the general case. **d)** No hints. You can still do this one without actually solving the first three subtasks.

mkilic

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