Linear Algebra
Parallel Lines and Linear Transformations

G-07

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We will show that after a linear transformation, parallel lines remain parallel. Here, a line is defined as follows:

Definition

A set L is a line in $\mathbb{R}^m$ if and only if there exists a vector $\mathbf{w} \in \mathbb{R}^m$, with $\mathbf{w} \neq \mathbf{0}$, such that

$$L = \{\lambda \mathbf{w} : \lambda \in \mathbb{R}\}.$$

But this definition of lines do not allow lines that do not go through 0. To change this we modify the definition a little bit (note that everything defined/solved/proposed in my notes are not official and not the primary course material)

A set L is a line in $\mathbb{R}^m$ if and only if there exist vectors $\mathbf{w} \in \mathbb{R}^m$ (with $\mathbf{w} \neq 0$) and $\mathbf{p} \in \mathbb{R}^m$ such that

$$L = \{\mathbf{p} + \lambda \mathbf{w} : \lambda \in \mathbb{R}\}.$$

Proof

Consider two parallel lines in $\mathbb{R}^m$, defined by the sets:

$$L_1 = \{\lambda \mathbf{w} : \lambda \in \mathbb{R}\}, \quad L_2 = \{\lambda \mathbf{w} + \mathbf{p} : \lambda \in \mathbb{R}\},$$

where $\mathbf{w} \in \mathbb{R}^m$ is the direction vector (with $\mathbf{w} \neq 0$), and $\mathbf{p} \in \mathbb{R}^m$ is a translation vector. (i.e. the vector between corresponding points on L_1 and L_2)

Let $\mathbf{T} : \mathbb{R}^m \to \mathbb{R}^m$ be a linear transformation. Applying $\mathbf{T}$ to the lines L_1 and L_2 gives (because of the homogeneity of the linear transformations $T(\lambda) = \lambda T(w)$):

$$\mathbf{T}(L_1) = \{\lambda \mathbf{T}(\mathbf{w}) : \lambda \in \mathbb{R}\}, \quad \mathbf{T}(L_2) = \{\lambda \mathbf{T}(\mathbf{w}) + \mathbf{T}(\mathbf{p}) : \lambda \in \mathbb{R}\}.$$

Conclusion

Both transformed lines have the same direction vector $\mathbf{T}(\mathbf{w})$, so the lines remain parallel after the transformation. Hence, linear transformations preserve the parallelism of lines.

$\square$

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