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G-07

HS 2024
22. IX. 2024

Linear Algebra
Affine Independence

We know from the lecture notes that all linear combinations of two vectors in 2D vector space gives us a line. But what about 3 vectors? Affine combinations of 3 vectors in 2D plane yield **the whole plane** but only if they are **affinely independent**. Otherwise the span is a line, just like with two vectors.

Definition:
A set of n vectors $v_1, \ldots, v_n$ are affinely independent if and only if the vectors

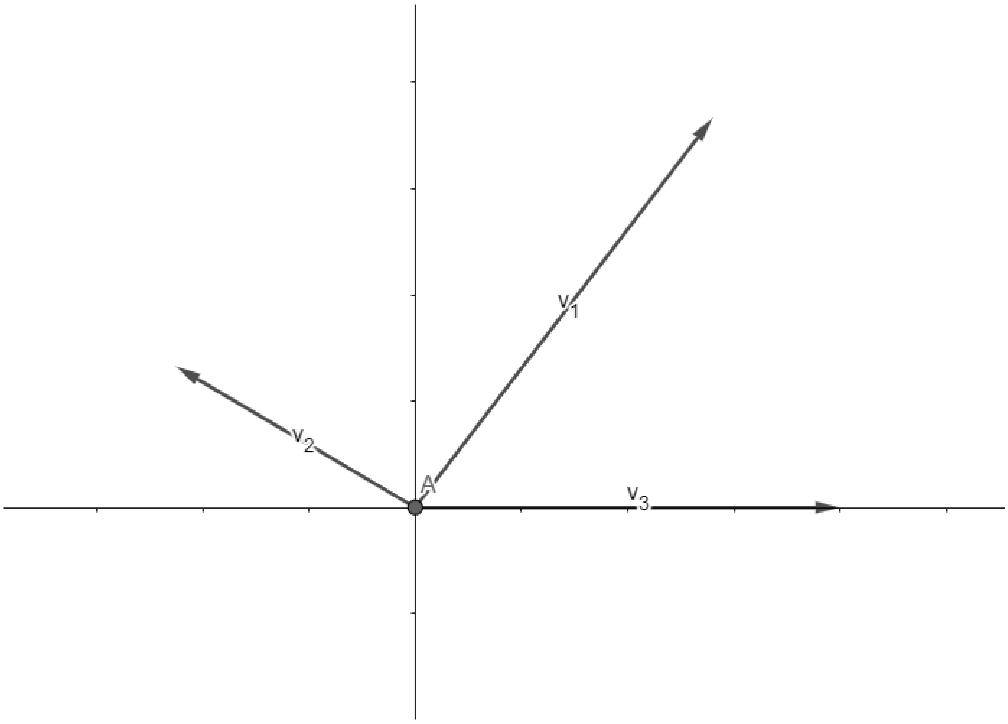
$$v_2 - v_1, v_3 - v_1, \ldots, v_n - v_1$$

are linearly independent.

First, note that v_0 is not special, we could make the same definition with the vectors $v_1 - v_2, v_3 - v_2, \ldots, v_n - v_2$ too. (You can check it for yourself if you want :)

This definition seems a bit odd, doesn't it? Let's understand why we define affine independence like this and prove that 3 affinely independent 2D vectors can span the whole plane.

1. Imagine 3 affinely independent vectors in 2D plane:



2. Write down all possible affine combinations. Let $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ and $v_1, v_2, v_3, \in \mathbb{R}^2$ be arbitrary

$$\lambda_1 \cdot v_1 + \lambda_2 \cdot v_2 + \lambda_3 \cdot v_3$$

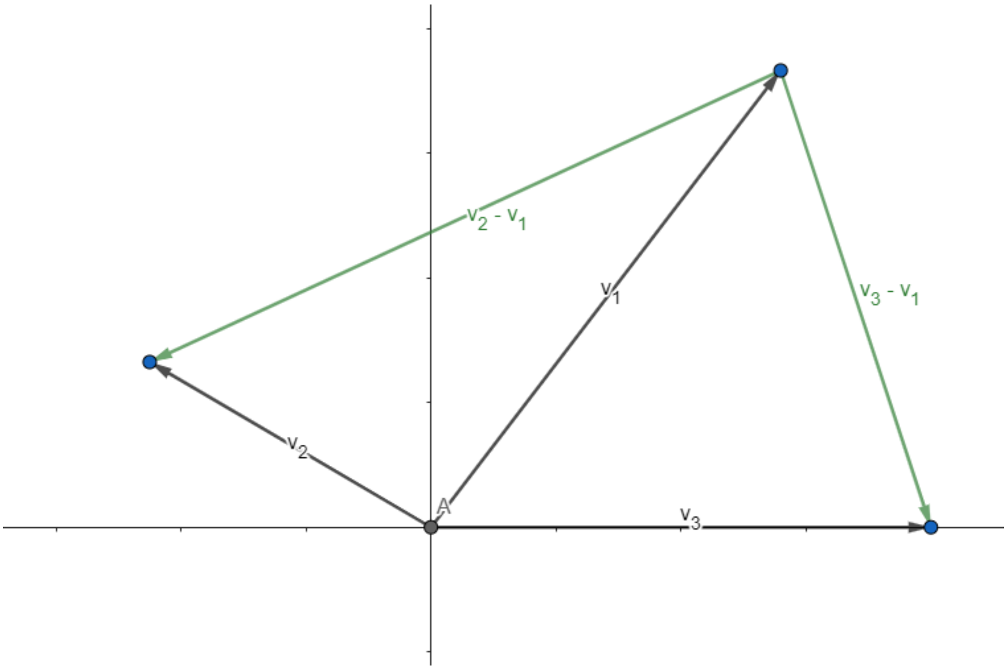
Where $\sum_{i=1}^{3} \lambda_i = \lambda_1 + \lambda_2 + \lambda_3 = 1$ (Definition of an affine combination).

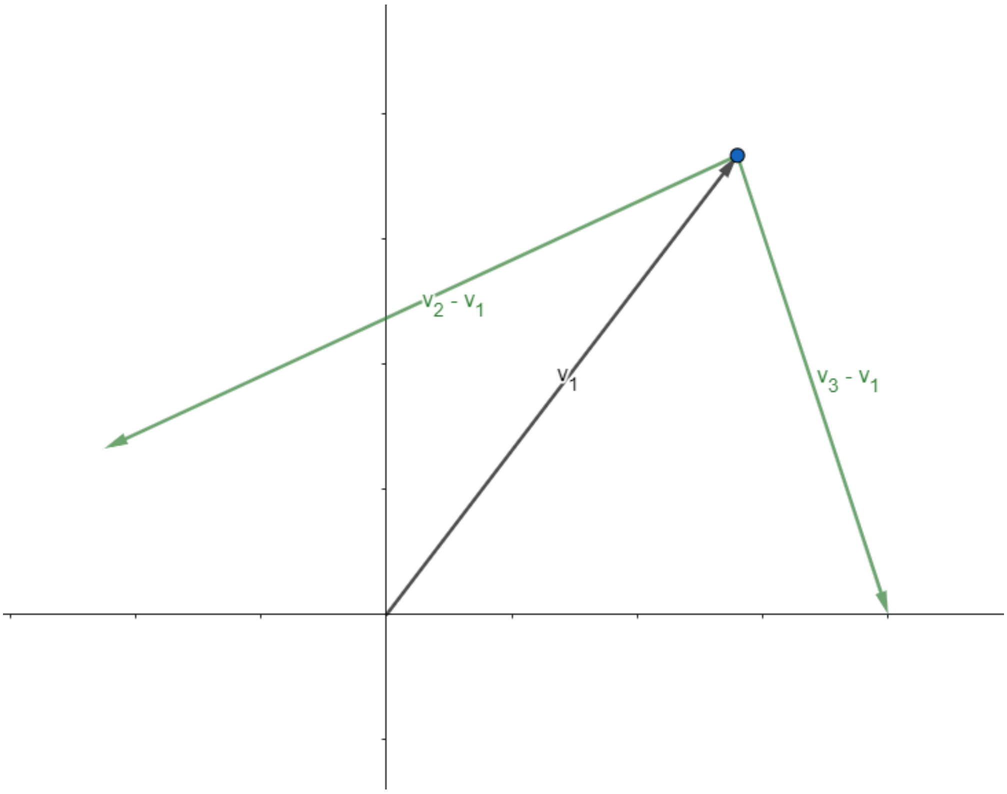
3. See that $\lambda_1 = 1 - \lambda_2 - \lambda_3$ because of the equality above.

4. Use 3. in the linear combinations in 2.

$$(1 - \lambda_2 - \lambda_3) \cdot v_1 + \lambda_2 \cdot v_2 + \lambda_3 \cdot v_3$$
$$\iff v_1 - \lambda_2 \cdot v_1 - \lambda_3 \cdot v_1 + \lambda_2 \cdot v_2 + \lambda_3 \cdot v_3$$
$$\iff v_1 + \lambda_2 \cdot (v_2 - v_1) + \lambda_3 \cdot (v_3 - v_1) \qquad (\star)$$

5. Interpreting $(\star)$ tells us to do two things: First, go to the point that v_1 takes you to and secondly, use scalar multiples of the difference vectors $(v_2 - v_1)$ and $(v_3 - v_1)$ to go somewhere else. Which is just a linear combination of the two vectors $(v_2 - v_1)$ and $(v_3 - v_1)$ without any limitation as to the values of λ_2 or λ_3 that takes v_1 as the start. This linear combination spans the whole space if and only if the two vectors are linearly independent, which corresponds to the affine independence. Since we assumed that the vectors are affinely independent in the beginning we can conclude that the span is the whole 2D plane. $\square$





A FEW CLOSING NOTES:

- In a vector space of dimension D, there can only be D linearly independent vectors but there can be $D+1$ affinely independent vectors.
- Any affine combination is a linear combination; therefore every affinely dependent set is linearly dependent. Conversely, every linearly independent set is affinely independent.

SOME USEFUL LINKS

Math Stack Exchange
Wikipedia (look for affine independence on the wiki page)
Slides from Seoul National University
ETHZ Institute of Theoretical Computer Science