Linear Algebra
Week 0

G-17

18 IX 2025

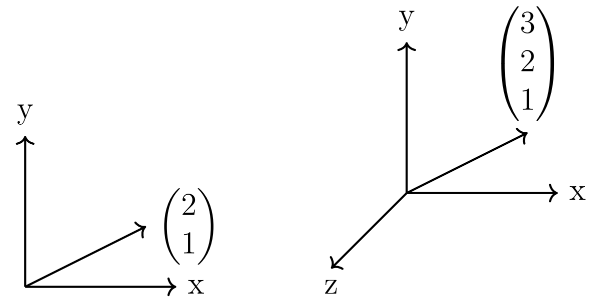
1 Vectors

Vectors are one of the main tools we use in linear algebra. For now they are defined as elements of $\mathbb{R}^m, m \in \mathbb{N}$ which represents the m dimensional space. So a vector $\mathbf{v} \in \mathbb{R}^m$ is a sequence of m real numbers.

This might bring up the question: How does $\mathbb{R}^0$ look like? In $\mathbb{R}^0$ we have only one element: the empty sequence () of real numbers.

1.1 Vectors in a Coordinate System

Thinking of vectors as a way to move around in the m dimensional space is a good way of understanding vectors. This might seem easy on a 2 dimensional example. However visualizing vectors and other objects we use in linear algebra like matrices and vector spaces is going to become more difficult as the number of dimensions increases.



(a) in 2 Dimensions

(b) in 3 Dimensions

2 Vector Addition and Scalar Multiplication

- Addition: $\mathbf{v}, \mathbf{u} \in \mathbb{R}^m : \mathbf{v} + \mathbf{u} = \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix} + \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix} = \begin{bmatrix} v_1 + u_1 \\ \vdots \\ v_m + u_m \end{bmatrix} \in \mathbb{R}^m$

- Scalar Multiplication: $\lambda \in \mathbb{R}, \mathbf{v} \in \mathbb{R}^m : \lambda \cdot \mathbf{v} = \lambda \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} \lambda \cdot v_1 \\ \lambda \cdot v_2 \\ \lambda \cdot v_3 \end{bmatrix} \in \mathbb{R}^m$

You can imagine the vector addition as putting the vectors and to end, in order to reach where you want to go. On the other hand, scalar multiplication is just deciding how many of the same vector you want to use. Multiplying by -1 changes the direction.

3 Linear Combinations

Combine vector addition and scalar multiplication.

Let $\lambda_1, \ldots, \lambda_m \in \mathbb{R}, \mathbf{v}_1, \ldots, \mathbf{v}_m \in \mathbb{R}^n$ Then:

$$\lambda_1 \cdot \mathbf{v}_1 + \cdots + \lambda_m \cdot \mathbf{v}_m$$

is a linear combination of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$.Note that the result is again a vector in the same vector space (*i.e.* the resulting vector has the same dimension). So *we can produce vectors using other vectors!*

A natural question is whether or not we can produce all vectors by using a given set of vectors. Think in 2 dimensions. It might be relatively easy to see that the vectors:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

can be used to produce all 2 dimensional vectors. If you want to write an arbitrary two dimensional vector

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

as a linear combination of these two vectors -where values of v_1 and v_2 are known to you- you write:

$$\lambda \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \text{ for } \lambda, \mu \in \mathbb{R}$$

and try to find the suited coefficients. In this case $\lambda = v_1$ and $\mu = v_2$.

4 Row & Column Pictures

We can distinguish between the column picture and the row picture when speaking of a linear combination. This example is from the lecture notes where you try to write an arbitrary vector

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathbb{R}^2$$

as the linear combination of

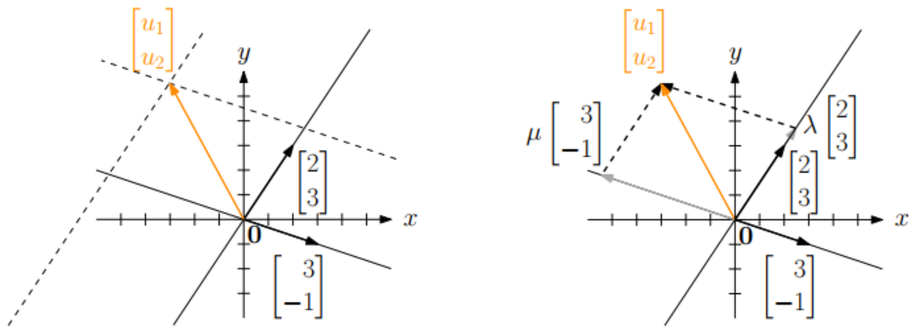
$$\mathbf{v} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

Then you do the same thing as above, try to find coefficients $\lambda, \mu \in \mathbb{R}$ such that

$$\lambda \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

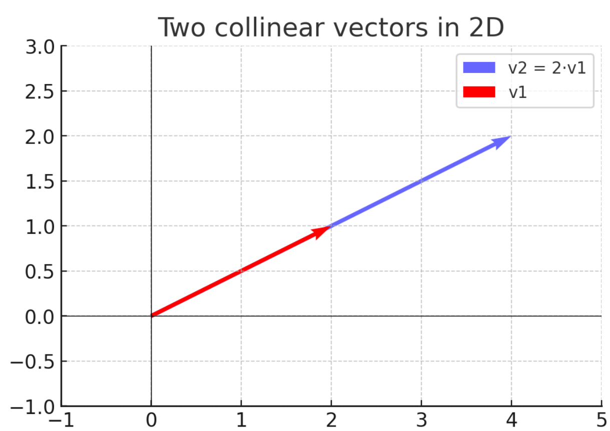
Now you can think of this linear combination in two ways:

The Column Picture



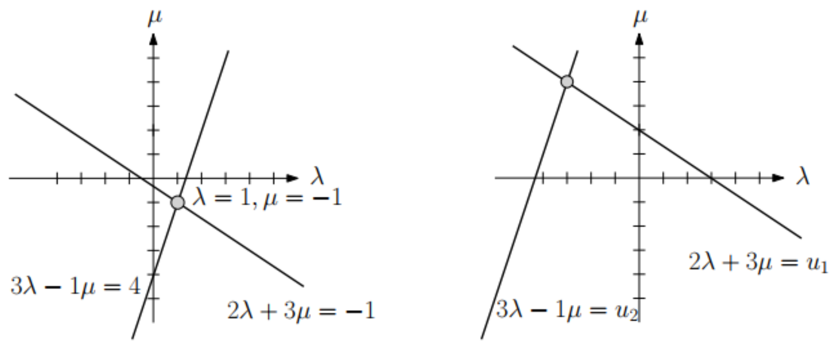
This approach tells you that you can build the vector **u** by adding the multiples of the vectors **v** and **w** end to end. You can use the vectors as building blocks or steps to get to the vector you want to reach. In the process of doing this, how many times you used which vector gives you λ and μ.

Now imagine that the two vectors were parallel to each other. When you put both to the origin they would be on a single line. This means no matter how you combine those vectors to get to some point in your coordinate system, you will stay on that line. Therefore in that scenario you cannot produce all vectors in your coordinate system using the ones you have.



Note that two vectors are parallel when they are a scalar multiple of each other. In this case we call them ***collinear***.

The Row Picture



The row picture actually serves the same purpose of finding λ and μ. However this time the focus is on the elements of the individual vectors. You write the linear combination in an extended form:

$$2\lambda + 3\mu = u_1$$
$$3\lambda - 1\mu = u_2$$

And focus on the lines that are represented by each equation. The intersection of these lines give you the values for λ and μ that satisfy both equations and hence the whole linear system of equations (LSE). We are going to deal wit LSE's extensively in the following weeks.

5 Affine, Conic, and Convex Combinations

We can observe different kinds of special linear combinations. For a linear combination of n vectors where$\lambda_1 \dots \lambda_n \in \mathbb{R}$ and $\mathbf{v}_1, \dots, \mathbf{v}_n \in \mathbb{R}^m$

$$\lambda_1 \mathbf{v}_1 + \cdots + \lambda_n \mathbf{v}_n$$

We say the linear combination is *affine, conic,* or *convex* if it satisfies the following corresponding conditions:

- Affine: $\lambda_1 + \cdots + \lambda_m = 1$
- Conic: $\forall i : \lambda_i \geq 0$
- Convex: Affine and Conic

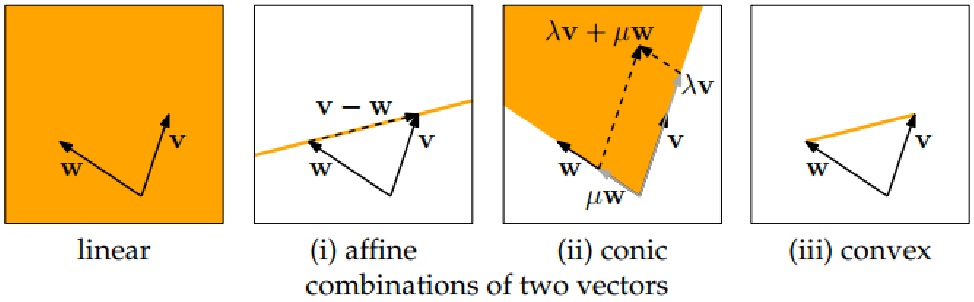


Figure 1.10: Two vectors and their combinations

5.1 Affine Combination of Two Vectors

We can take a closer look to affine linear combinations to better understand why they form a line for two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in $\mathbb{R}^2$.

$\lambda_1 \cdot \mathbf{v}_1 + \lambda_2 \cdot \mathbf{v}_2$ (since $\lambda_1 + \lambda_2 = 1$)
$= \lambda_1 \cdot \mathbf{v}_1 + (1 - \lambda_1) \cdot \mathbf{v}_2$
$= \lambda_1 \cdot \mathbf{v}_1 + 1 \cdot \mathbf{v}_2 - \lambda_1 \cdot \mathbf{v}_2$
$= \mathbf{v}_2 + \lambda_2(\mathbf{v}_1 - \mathbf{v}_2)$

The last line of equation tells us to go to v_2 first and then use any scalar multiple of the vector $v_1 - v_2$ to go somewhere on the line that goes through v_2 and is parallel to $v_1 - v_2$.

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