Linear Algebra
What does failure actually mean in Gaussian Elimination?

G-07

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This pdf *What does failure actually mean?* is not strictly formal and not exactly relevant for this week (week 4) but it provides good intuition which is going to come in handy in the following weeks.

What does failure actually mean?

We fail when we have a pivot element equal to 0 and can not get out of this situation by applying row exchange *i.e.* all entries below the current pivot are also 0. Two examples are:

$$\begin{bmatrix} 2 & 3 & 4 \\ \color{blue}{0} & \color{red}{\mathbf{0}} & 6 \\ \color{blue}{0} & \color{red}{0} & 13 \end{bmatrix}$$

(a) Failure in 2nd row

$$\begin{bmatrix} 2 & 3 & 4 \\ \color{blue}{0} & 5 & 6 \\ \color{blue}{0} & \color{blue}{0} & \color{red}{\mathbf{0}} \end{bmatrix}$$

(b) Failure in the last row

We can demonstrate two different situations using these two examples:

1. When we do not have any solution in the sense that the solution does not exist
2. When we have infinitely many solutions.

It is not important at which row we fail for the following examples. These two examples where Gauss Elimination fails in the lecture notes, this is why they are here as examples.

- Let's observe the example "*(a) Failure in 2nd row*" where our U is the matrix in figure (a). Note that even though it has 0 elements on the diagonal it is still upper triangular per definition.

  For the following let $\mathbf{c} = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$ be a vector so that $U\mathbf{x} = \mathbf{c}$ We just choose some fix $\mathbf{c}$ for demonstration purposes.

  The linear system of equations represented by our matrix is:

$$\begin{aligned} 2x_1 + 3x_2 + 4x_3 &= 3 \\ 6x_3 &= 4 \\ 13x_3 &= 2 \end{aligned}$$

  But this leads us to two equations $x_3 = \frac{4}{6}$ and $x_3 = \frac{2}{13}$ which can not hold at the same time. **This linear system of equations does not have any solutions**.

- Now, let us observe the example "*(b) Failure in the last row*" where our U is the matrix in figure (b). Note that even though it has 0 elements on the diagonal it is still upper triangular per definition.

  For the following let $\mathbf{c} = \begin{bmatrix} 6 \\ 5 \\ 0 \end{bmatrix}$ be a vector so that $U\mathbf{x} = \mathbf{c}$ We just choose some fix $\mathbf{c}$ for demonstration purposes.

  The linear system of equations represented by our matrix is:

$$\begin{aligned} 2x_1 + 3x_2 + 4x_3 &= 6 \\ 5x_2 + 6x_3 &= 5 \\ 0 &= 0 \end{aligned}$$

  We are lucky: we have $0 = 0$ in the last row. If the 3rd element of b was not 0, then this LSE would not have any solutions as in the previous example. Having said that, how do we solve this LSE? You can express everything in terms of one variable: x_3. If you do this our solution is:

$$\boxed{x_1 = \frac{15 - 2t}{10}, \quad x_2 = \frac{5 - 6t}{5}, \quad x_3 = t \quad (t \in \mathbb{R})}$$

  This is valid for all $t \in \mathbb{R}$. Hence for each value of t we have another set of x_1, x_2, x_3 that satisfies our equations. **We have infinitely many solutions!**

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