Linear Algebra
In-Class Exercise Week 2

G-17

2 X 2025

1. Rank of a matrix (in-class) (★★☆)

Let $m \in \mathbb{N}_{\geq 2}$ be arbitrary and consider the $m \times m$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{bmatrix}$$

with $a_{ij} = i + j$ for all $i, j \in \{1, 2, \dots, m\}$. Determine the rank of A. You need to justify your answer.

1 Intuition

It is always good practice to handle the situation at a concrete example to gain intuition about the task.

Let $m = 4$. Then our matrix A is:

$$\begin{bmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 5 & 6 & 7 & 8 \end{bmatrix}$$

In order to determine the rank we should look at the columns of A since rank is defined as the number of independent columns. A quick scan beginning from the first column shows us that the first two columns are independent. Okay but how do we see this?

Let $\mathbf{v}_1, \mathbf{v}_2$ be the first two column vectors. We know that two vectors are linearly dependent if one of them can be written as a scalar multiple of the other. This means for $\mathbf{v}_1, \mathbf{v}_2$ to be dependent there must exist a $\lambda \in \mathbb{R}$ that satisfies: $\lambda \cdot \mathbf{v}_1 = \mathbf{v}_2$. However, this would mean that $\lambda \cdot 2 = 3$ and $\lambda \cdot 3 = 4$ (considering only the first two elements of $\mathbf{v}_1$ and $\mathbf{v}_2$). To satisfy these equations λ has to be equal to both $\frac{3}{2}$ and $\frac{4}{3}$. This gives us the contradiction we want and prove that $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent.

To see that the other vectors are dependent, note that $\mathbf{v}_2 - \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

If you add the vector $\mathbf{v}_2 - \mathbf{v}_1$ to the vector $\mathbf{v}_2$ you get the third column $\mathbf{v}_2 + (\mathbf{v}_2 - \mathbf{v}_1) = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \end{pmatrix}$ Analogously you can also write the fourth column as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. This is why only independent vectors are $\mathbf{v}_1$ and $\mathbf{v}_2$, so $\mathbf{rank}(A) = 2$.

2 Solution

Above we had an example, but not a proof. Now we have to argue for all m's formally. Note that this solution is slightly different than the master solution.

Let $m \in \mathbb{N}_{\geq 2}$ be arbitrary and $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m \in \mathbb{R}^m$ such that $A = \begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_m \\ | & | & & | \end{bmatrix}$

Since $\mathbf{v}_1$ is not equal to the zero vector, the rank is at least 1. Besides there exists no $\lambda \in \mathbb{R}$ s.t. $\lambda \cdot \mathbf{v}_1 = \mathbf{v}_2$ since this λ would have to satisfy

$$2 \cdot \lambda = a_{11} \cdot \lambda = a_{12} = 3$$

by looking at the first coordinates and

$$3 \cdot \lambda = a_{21} \cdot \lambda = a_{22} = 4$$

by looking at the second coordinates. This would yield $\frac{3}{2} = \lambda = \frac{4}{3}$, an equation that does not hold. Therefore $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent and A has rank at least two. (So far the same as the intuition.)

Now see that $\mathbf{v}_2 - \mathbf{v}_1 = \begin{pmatrix} a_{21} \\ a_{22} \\ \vdots \\ a_{2m} \end{pmatrix} - \begin{pmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{1m} \end{pmatrix} = \begin{pmatrix} (2+1)-(1+1) \\ (2+2)-(2+1) \\ \vdots \\ (2+m)-(1+m) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$

A more formal notation is writing $\mathbf{v}_2 - \mathbf{v}_1 = \big((2+i) - (1+i)\big)_{i=1}^{m} = \big(1\big)_{i=1}^{m}$

Now that we managed to write the vector $\mathbf{1} \in \mathbb{R}^m$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, we can use it to create the other columns of A by adding it to $\mathbf{v}_1$ just as we did in the intuition part.

First we calculate $\mathbf{v}_j - \mathbf{v}_1$ for any $3 \leq j \leq m$:

$$\mathbf{v}_j - \mathbf{v}_1 = \big(a_{ij} - a_{i1}\big)_{i=1}^{m} = \big((i+j) - (i+1)\big)_{i=1}^{m} = \big((j-1)\big)_{i=1}^{m}$$

This means that the difference between the column j of A and the first column of A is equal to $\begin{pmatrix} j-1 \\ j-1 \\ \vdots \\ j-1 \end{pmatrix} \in \mathbb{R}^m$.

To conclude the proof we combine our findings so far: for any arbitrary j with $3 \leq j \leq m$:

$$\mathbf{v}_j - \mathbf{v}_1 = \begin{pmatrix} j-1 \\ j-1 \\ \vdots \\ j-1 \end{pmatrix} = (j-1) \cdot \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = (j-1) \cdot (\mathbf{v}_2 - \mathbf{v}_1)$$

$$\Rightarrow \mathbf{v}_j - \mathbf{v}_1 = (j-1) \cdot (\mathbf{v}_2 - \mathbf{v}_1)$$

$$\Rightarrow \mathbf{v}_j - \mathbf{v}_1 = j \cdot \mathbf{v}_2 - j \cdot \mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_1$$

$$\Rightarrow \mathbf{v}_j - \mathbf{v}_1 = (j-1) \cdot \mathbf{v}_2 + (1-j) \cdot \mathbf{v}_1$$

$$\Rightarrow \mathbf{v}_j = (j-1) \cdot \mathbf{v}_2 + (2-j) \cdot \mathbf{v}_1$$

We showed that any column of A can be written as a linear combination of the first two columns $\mathbf{v}_1$ and $\mathbf{v}_2$ which are themselves linearly independent from each other. Therefore the matrix A has only 2 independent columns. We conclude $\mathbf{rank}(A) = 2$ for all m.

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