Linear Algebra
In-Class Exercise Week 3

G-07

12 X 2024

1. Linear transformations (in-class) (★☆☆)

a) Let $n \in \mathbb{N}^+$. Consider the function $T : \mathbb{R}^n \to \mathbb{R}$ defined by

$$T(\mathbf{x}) := \sum_{k=1}^{n} k x_k$$

for all $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}^\top \in \mathbb{R}^n$. Prove that T is a linear transformation.

b) Let $n \in \mathbb{N}^+$ with $n \geq 2$ be arbitrary. Consider the function $T : \mathbb{R}^n \to \mathbb{R}$ defined by

$$T(\mathbf{x}) := \sum_{k=1}^{n} (x_k)^k$$

for all $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}^\top \in \mathbb{R}^n$. Is T a linear transformation?

Definition 2.27 (Linear transformation) *Let $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a function from $\mathbb{R}^n$ to $\mathbb{R}^m$. T is called a linear transformation if the following two statements hold for all $x, y \in \mathbb{R}^n$ and all $\lambda \in \mathbb{R}$.*

(i) $T(x + y) = T(x) + T(y)$ and

(ii) $T(\lambda x) = \lambda T(x)$.

So we have to prove these two statements to prove that a function is a linear transformation, whereas we show that at least one of these statements does not hold true for a function to prove that it is not a linear transformation.

1 Solution

1.1 a)

In this subtask we verify the two statements above for the given T. Let $x, y \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$ be arbitrary.

Additivity:

$$T(\mathbf{x}+\mathbf{y}) = \sum_{k=1}^{n} k(x_k+y_k) = \sum_{k=1}^{n}(kx_k+ky_k) = \sum_{k=1}^{n} kx_k+\sum_{k=1}^{n} ky_k = T(x)+T(y)$$

Homogeneity:

$$T(\lambda\mathbf{x}) = \sum_{k=1}^{n} \lambda x_k = \lambda \sum_{k=1}^{n} x_k = \lambda T(x)$$

This is the complete proof that T is a linear transformation.

□

1.2 b)

No T is not a linear transformation. If you want to see this by eyeballing, you can convince yourself by intuitively considering that $(a+b)^k \neq a^k + b^k$ in general for real numbers.

But first note that this transformation is actually linear for $n = 1$. Because in 1 dimension the function T is nothing else but the identity function, which is a linear transformation.

So let $n \geq 2$:

Since we want to prove that T is not a linear transformation, it is sufficient that we find a counter example. In the master solution there is a counter example to the homogeneity, therefore here is a counterexample to additivity (which can also be seen as a counterexample to homogeneity if you play around a little on this proof):

Let $\mathbf{e}_n \in \mathbb{R}^n$ with $\mathbf{x} = \lambda \mathbf{e}_n$ and $\mathbf{y} = \mu \mathbf{e}_n$ Also let λ and μ be positive integers. We are free to choose λ and μ as we wish because we are just providing a counter *example*.

$$i.e.\ \mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \lambda \end{bmatrix} \text{ and } \mathbf{y} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \mu \end{bmatrix}$$

Now calculate:

$T(\mathbf{x}) = \sum_{k=1}^{n} (x_k)^k = 0^1 + 0^2 + \cdots + \lambda^n = \lambda^n$

$T(\mathbf{y}) = \sum_{k=1}^{n} (y_k)^k = 0^1 + 0^2 + \cdots + \mu^n = \mu^n$

$T(\mathbf{x}+\mathbf{y}) = \sum_{k=1}^{n} (x_k + y_k)^k = (0+0)^1 + (0+0)^2 + \cdots + (\lambda+\mu)^n = (\lambda+\mu)^n$

For T to be a linear transformation we must have $T(x) + T(y) = T(x+y)$ If we fill in this equation with our results from above we get:

$$\lambda^n + \mu^n \stackrel{!}{=} (\lambda + \mu)^n$$

Does this seem familiar to you? The equation is never true for $n > 2$ and for positive λ and μ. This fact follows from Fermat's Last Theorem. Fermat did not provide a proof for his theorem and it was not before more than 300 years that in 1995 world of mathematics (Sir Andrew Wiles) finally came up with a proof. Now we can use this incredibly powerful heritage of mathematics to prove our claim. Note that for $n = 2$ we can argue using Pythagorean Theorem by choosing λ and μ accordingly *e.g.* 1 and 2. $1^2 + 2^2 = 5 \neq 9 = 3^2 = (1+2)^2$. Hence we have proven that

$$T(x+y) \neq T(x) + T(y)$$

which shows us that the function T is not a linear transformation for $n \geq 2$.

$\square$

**Note that you are not required to use cool theorems to prove your answer in this question. Since we are giving a counterexample it is enough to choose real λ and μ, and argue that e.g. $5^n + 6^n \neq 11^n$.*

mkilic

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