Linear Algebra
In-Class Exercise Week 2

G-07

08 X 2024

1. Rank of a matrix (in-class) (★★☆)

Let $m \in \mathbb{N}_{\geq 2}$ be arbitrary and consider the $m \times m$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{bmatrix}$$

with $a_{ij} = i + j$ for all $i, j \in \{1, 2, \dots, m\}$. Determine the rank of A. You need to justify your answer.

1 Intuition

It is always good practice to handle the situation at a concrete example to gain intuition about the task.

Let $m = 4$. Then our matrix A is:

$$\begin{bmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 5 & 6 & 7 & 8 \end{bmatrix}$$

In order to determine the rank we should look at the columns of A since rank is defined as the number of independent columns. A quick scan beginning from the first column shows us that the first two columns are independent. Okay but how do we see this?

Let v_1, v_2 be the first two column vectors. We know that two vectors are linearly dependent if one of them can be written as a scalar multiple of the other. This means for v_1, v_2 to be dependent there must exist a $\lambda \in \mathbb{R}$ that satisfies: $\lambda \cdot v_1 = v_2$. However, this would mean that $\lambda \cdot 2 = 3$ and $\lambda \cdot 3 = 4$ (considering only the first two elements of v_1 and v_2). To satisfy these equations λ has to be equal to both $\frac{3}{2}$ and $\frac{4}{3}$. This gives us the contradiction we want and prove that v_1 and v_2 are linearly independent.

To see that the other vectors are dependent, note that $v_2 - v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

If you add the vector $v_2 - v_1$ to the vector v_2 you get the third column $v_2 + (v_2 - v_1) = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \\ 7 \end{bmatrix}$ Analogously you can also write the fourth column as a linear combination of v_1 and v_2. This is why only independent vectors are v_1 and v_2, so $\mathbf{rank}(A) = 2$.

2 Solution

Above we had an example, but not a proof. Now we have to argue for all m's formally. Note that this solution is slightly different than the master solution.

Let $m \in \mathbb{N}_{\geq 2}$ be arbitrary and $v_1, v_2, \ldots, v_m \in \mathbb{R}^m$ such that $A = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \ldots & v_m \\ | & | & & | \end{bmatrix}$

Since v_1 is not equal to the zero vector, the rank is at least 1. Besides there exists no $\lambda \in \mathbb{R}$ s.t. $\lambda \cdot v_1 = v_2$ since this λ would have to satisfy

$$2 \cdot \lambda = a_{11} \cdot \lambda = a_{12} = 3$$

by looking at the first coordinates and

$$3 \cdot \lambda = a_{21} \cdot \lambda = a_{22} = 4$$

by looking at the second coordinates. This would yield $\frac{3}{2} = \lambda = \frac{4}{3}$, an equation that does not hold. Therefore v_1 and v_2 are linearly independent and A has rank at least two. (So far the same as the intuition.)

Now see that $v_2 - v_1 = \begin{bmatrix} a_{21} \\ a_{22} \\ \vdots \\ a_{2m} \end{bmatrix} - \begin{bmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{1m} \end{bmatrix} = \begin{bmatrix} (2+1)-(1+1) \\ (2+2)-(2+1) \\ \vdots \\ (2+m)-(1+m) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$

A more formal notation is writing $v_2 - v_1 = \big[(2+i)-(1+i)\big]_{i=1}^{m} = \big[1\big]_{i=1}^{m}$

Now that we managed to write the vector $\mathbf{1} \in \mathbb{R}^m$ as a linear combination of v_1 and v_2, we can use it to create the other columns of A by adding it to v_1 just as we did in the intuition part.

First we calculate $v_j - v_1$ for any $2 \leq j \leq m$:

$$v_j - v_1 = \big[a_{ij} - a_{i1}\big]_{i=1}^{m} = \big[(i+j)-(i+1)\big]_{i=1}^{m} = \big[(j-1)\big]_{i=1}^{m}$$

This means that the difference between the column j of A and the first column of A is equal to $\begin{bmatrix} j-1 \\ j-1 \\ \vdots \\ j-1 \end{bmatrix} \in \mathbb{R}^m$.

To conclude the proof we combine our findings so far: for any arbitrary j with $2 \leq j \leq m$:

$$v_j - v_1 = \begin{bmatrix} j-1 \\ j-1 \\ \vdots \\ j-1 \end{bmatrix} = (j-1) \cdot \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = (j-1) \cdot (v_2 - v_1)$$

$\Rightarrow v_j - v_1 = (j-1) \cdot (v_2 - v_1)$

$\Rightarrow v_j - v_1 = j \cdot v_2 - j \cdot v_1 - v_2 + v_1$

$\Rightarrow v_j - v_1 = (j-1) \cdot v_2 + (1-j) \cdot v_1$

$\Rightarrow v_j = (j-1) \cdot v_2 + (2-j) \cdot v_1$

We showed that any column of A can be written as a linear combination of the first two columns v_1 and v_2 which are themselves linearly independent from each other. Therefore the matrix A has only 2 independent columns. We conclude **rank**$(A) = 2$ for all m.

$\square$

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