Linear Algebra
Calculating the Inverse
Using Gauss Elimination

G-17

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1 Algorithm: Calculating the Inverse

We can use Gauss elimination with back substitution to calculate the inverse of a matrix. Let's put the problem of calculating the inverse in terms of solving LSE's. We want to find the inverse of a matrix $A \in \mathbb{R}^{m\times m}$. Call $A^{-1} = B$ for sake of simplicity in notation. We look for $B \in \mathbb{R}^{m\times m}$ with columns $\mathbf{b}_1, \ldots, \mathbf{b}_m \in \mathbb{R}^m$ such that:

$$AB = I$$

Taking a closer look into the matrix multiplication, we can formulate this expression as follows:

$$\begin{bmatrix} | & | & & | \\ A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_m \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_m \\ | & | & & | \end{bmatrix}$$

The equality of two matrices means their columns must match. So we want the columns of B to satisfy:

$$\begin{aligned} A\mathbf{b}_1 =& \mathbf{e}_1 \\ A\mathbf{b}_2 =& \mathbf{e}_2 \\ &\vdots \\ A\mathbf{b}_m =& \mathbf{e}_m \end{aligned}$$

There we have m linear systems of equations, all of which we can try to solve using Gauss elimination with back substitution. If Gauss elimination fails we know by ***Theorem 3.7*** and ***Theorem 3.8*** that the matrix A has linearly dependent columns and hence is not invertible. Therefore we can assume that Gauss elimination succeeds for the following computations.

Applying Gauss elimination transforms A into an upper triangular form and all $\mathbf{e}_i$'s are also modified during the process. For upper triangular U we have:

$$\begin{aligned} U\mathbf{b}_1 =& \mathbf{c}_1 \\ U\mathbf{b}_2 =& \mathbf{c}_2 \\ &\vdots \\ U\mathbf{b}_m =& \mathbf{c}_m \end{aligned}$$

Now we can do back substitution to find the individual columns $\mathbf{b}_i$. We will have calculated the matrix once we have all the columns.

2 Runtime Optimization: m right hand sides

If you try to do the Gauss elimination (that has runtime $O(m^3)$) step from scratch for all m equations we get a runtime of $O(m^4)$. We do not want this. You probably agree that calculating the U from scratch is not necessary since U is determined only by the matrix A. What we can do to improve this, is to apply the algorithm for m right hand side vectors.

We usually write the matrix in augmented form as $[A|\mathbf{y}]$ when we try to solve one linear system of equation $A\mathbf{x} = \mathbf{y}$. We do the same here, but for all possible right hand sides. To continue with the example from above we have:

$$\left[\begin{array}{cccc|cccc} | & | & & | & | & | & & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_m & \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_m \\ | & | & & | & | & | & & | \end{array}\right].$$

Now you can apply row subtractions and row swaps to A and at the same time to all m right hand side vectors. In the end you have:

$$\left[\begin{array}{cccc|cccc} | & | & & | & | & | & & | \\ \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_m & \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_m \\ | & | & & | & | & | & & | \end{array}\right].$$

where u_i's are the columns of the upper triangular matrix U and c_j's are the right hand side vectors after elimination. This operation takes $O(m^3)$ time since we apply the elimination to all right hand sides at the same time. To convince yourself to this fact you can read section ***3.2.6*** from the lecture notes.

3 Example

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$ then we have:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 5 & 6 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 - 5R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & -4 & -15 & -5 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow R_3 + 4R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & -5 & 4 & 1 \end{array}\right]$$

Now we have our upper triangular matrix U on the left side and the vectors $\mathbf{c}_j$ from before on the right side. We can do back substitution 3 times to get the inverse:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{b}_1 = \begin{pmatrix} 1 \\ 0 \\ -5 \end{pmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{b}_2 = \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{b}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

The back substitutions yield:

$$B = \begin{bmatrix} | & | & | \\ \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \\ | & | & | \end{bmatrix} = A^{-1} = \begin{bmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{bmatrix}.$$