Linear Algebra
In Class Exercise Week 1

G-17

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1. Lines in $\mathbb{R}^m$ (in-class) (★★☆)

a) Let $\mathbf{0} \in \mathbb{R}^m$ denote the vector whose entries are all zero. We say that a set L is a line through the origin in $\mathbb{R}^m$ if and only if there exists $\mathbf{w} \in \mathbb{R}^m$ with $\mathbf{w} \neq \mathbf{0}$ such that $L = \{\lambda \mathbf{w} : \lambda \in \mathbb{R}\}$. Let now L be a line through the origin in $\mathbb{R}^m$ and let $\mathbf{u}$ be an arbitrary non-zero element of L. Prove that $L = \{\lambda \mathbf{u} : \lambda \in \mathbb{R}\}$.

b) For two lines through the origin L_1 and L_2 in $\mathbb{R}^m$, prove that we have either $L_1 \cap L_2 = \{\mathbf{0}\}$ or $L_1 \cap L_2 = L_1 = L_2$.

c) Consider an arbitrary line through the origin L in $\mathbb{R}^2$. Prove that L is a hyperplane, i.e. find a vector $\mathbf{d} \neq \mathbf{0}$ such that

$$L = \{\mathbf{v} \in \mathbb{R}^2 : \mathbf{v} \cdot \mathbf{d} = 0\}.$$

1 Solution

The question defines lines as *sets*. In particular sets of linear combinations of one vector w, which is only scalar multiples of w. This means that in order to prove $L = \{\lambda \cdot u : \lambda \in \mathbb{R}\}$ we have to show the equality of two sets.

How to show the equality of two sets?

In general when we want to show the equality of two sets A, B, we show $A \subseteq B$ **and** $B \subseteq A$. If the two conditions hold together then we can conclude that the two sets are equal.

The following pages have the necessary steps of the proofs and also some explanations to make it clear.

a)

- We have $\mathbf{u} \in L$ and $\mathbf{u} \neq \mathbf{0}$ given. Since every element in L can be written as a scalar multiple of $\mathbf{w}$ we can write (for some $\lambda_{\mathbf{u}} \in \mathbb{R}$ where $\lambda_{\mathbf{u}} \neq 0$ since $\mathbf{u} \neq \mathbf{0}$):
$$\mathbf{u} = \lambda_{\mathbf{u}} \cdot \mathbf{w}$$
- Now we show that any element in L is also in $\{\lambda \cdot \mathbf{u} : \lambda \in \mathbb{R}\}$ *i.e.* $L \subseteq \{\lambda \cdot \mathbf{u} : \lambda \in \mathbb{R}\}$
- Let $\mathbf{v} \in L$ be arbitrary. Since $\mathbf{v}$ is on our line L, by the definition of L we must have some $\lambda_{\mathbf{v}} \in \mathbb{R}$ such that: $\mathbf{v} = \lambda_{\mathbf{v}} \cdot \mathbf{w}$
- Now write $\mathbf{w}$ in terms of $\mathbf{u}$ as $\mathbf{w} = \frac{1}{\lambda_{\mathbf{u}}} \cdot \mathbf{u}$. This gives us $\mathbf{v} = \frac{\lambda_{\mathbf{v}}}{\lambda_{\mathbf{u}}} \cdot \mathbf{u}$ Since we can write $\mathbf{v}$ as a scalar multiple of $\mathbf{u}$ this means $\mathbf{v} \in \{\lambda \cdot \mathbf{u} : \lambda \in \mathbb{R}\}$
- Because $\mathbf{v}$ was arbitrary, *i.e.* we were observing any element from the set L, the previous point proves that $L \subseteq \{\lambda \cdot \mathbf{u} : \lambda \in \mathbb{R}\}$
- Now we prove the other direction: $\{\lambda \cdot \mathbf{u} : \lambda \in \mathbb{R}\} \subseteq L$ For this take an arbitrary vector $\mathbf{v}' \in \{\lambda \cdot \mathbf{u} : \lambda \in \mathbb{R}\}$. We know $\mathbf{v}' = \lambda_{\mathbf{v}'} \cdot \mathbf{u}$
- We also know that $\mathbf{u} = \lambda_{\mathbf{u}} \cdot \mathbf{w}$ from the first point.
- Combining the last two points gives us $\mathbf{v}' = \lambda_{\mathbf{v}'} \cdot \mathbf{u} = \lambda_{\mathbf{v}'} \lambda_{\mathbf{u}} w$ Since we can write $\mathbf{v}'$ as a linear multiple of $\mathbf{w}$, we can conclude that $\mathbf{v}' \in L$
- Since v' was arbitrary, *i.e.* we were observing any element from the set $\{\lambda \cdot \mathbf{u} : \lambda \in \mathbb{R}\}$, the previous point proves that $\{\lambda \cdot \mathbf{u} : \lambda \in \mathbb{R}\} \subseteq L$
- The subset relation in both ways prove that $L = \{\lambda \cdot \mathbf{u} : \lambda \in \mathbb{R}\}$

$\square$

b)

- Let L_1 and L_2 two lines in $\mathbb{R}^m$. This means by definition of a line that there exist $\mathbf{w}_1, \mathbf{w}_2 \in \mathbb{R}$ such that:

$$L_1 = \{\lambda \cdot \mathbf{w}_1 : \lambda \in \mathbb{R}\} \text{ and } L_2 = \{\lambda \cdot \mathbf{w}_2 : \lambda \in \mathbb{R}\}$$

- We know $\mathbf{0} \in L_1 \cap L_2$ this means $\mathbf{0} \in L_1$ and $\mathbf{0} \in L_2$ because we have $\mathbf{0} = 0 \cdot \mathbf{w}_1 = 0 \cdot \mathbf{w}_2$ (we can choose $\lambda = 0$ for L_1 and L_2)

- Now assume $L_1 \cap L_2 \neq \{\mathbf{0}\}$. In words let's assume that there are more than just the vector $\mathbf{0}$ in the intersection of the two lines.

- Since the intersection is not empty by our assumption we have a non-zero vector $\mathbf{u} \in L_1 \cap L_2$. This means $\mathbf{u} \in L_1$ *and* $\mathbf{u} \in L_2$.

- Using the sub-task *a)* we can see that $\mathbf{u} \in L_1$ means L_1 can be written as $L_1 = \{\lambda \cdot \mathbf{u} : \lambda \in \mathbb{R}\}$ and $\mathbf{u} \in L_2$ means L_2 can also be written as $L_2 = \{\lambda \cdot \mathbf{u} : \lambda \in \mathbb{R}\}$

- This means, if the two lines share a non-zero element, they are exactly the same lines, *i.e.* $L_1 = L_2$.

$\square$

c)

- Since L is a line there must be a vector $\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in \mathbb{R}^2$ such that $L = \{\lambda \mathbf{w} : \lambda \in \mathbb{R}\}$

- We want $\mathbf{w} \cdot \mathbf{d} = w_1 d_1 + w_2 d_2 = 0$. Because then $\mathbf{d}$ is orthogonal to $\mathbf{w}$ and thus it is orthogonal to all vectors in L.

- Now we *guess* $\mathbf{d}$ the most intuitive way. If we choose $d_1 = -w_2$ and $d_2 = w_1$ then we have $w_1 d_1 + w_2 d_2 = -w_1 w_2 + w_2 w_1 = 0$.

- We now prove (for our selection of $\mathbf{d}$ from the previous point)

$$L = \{\mathbf{v} \in \mathbb{R}^2 : \mathbf{v} \cdot \mathbf{d} = 0\}$$

- $\subseteq$: Let $\mathbf{u} = \lambda_{\mathbf{u}}\mathbf{w}$ be an arbitrary element of L. Then we have

$$\mathbf{u} \cdot \mathbf{v} = (\lambda_{\mathbf{u}}\mathbf{u}) \cdot \mathbf{d} = \lambda_{\mathbf{u}}(\mathbf{u} \cdot \mathbf{d}) = 0$$

  Hence we can say that $\mathbf{u}$ is also an element of the hyperplane $\{\mathbf{v} \in \mathbb{R}^2 : \mathbf{v} \cdot \mathbf{d} = 0\}$. We have proven the $\subseteq$ direction.

- $\supseteq$: Consider a vector $\mathbf{u} \in \{\mathbf{v} \in \mathbb{R}^2 : \mathbf{v} \cdot \mathbf{d} = 0\}$ to be an arbitrary element of the hyperplane. Per definition of the hyperplane we have

$$\mathbf{u} \cdot \mathbf{d} = -u_1 w_2 + u_2 w_1 = 0 \quad (\star)$$

- We note that since L defines a line and by the definition of a line the spanning vector $\mathbf{w}$ cannot be the zero vector, we have either $w_1 \neq 0$ or $w_2 \neq 0$. Now for these two cases we proceed separately. We do this distinction because we want to divide by w_1 or w_2 and we have to make sure that we do not divide by 0.

- $w_1 \neq 0$: Using $(\star)$ we can write $u_2 = \frac{w_2}{w_1}u_1$. We look for a $\lambda \in \mathbb{R}$ such that we have $\mathbf{u} = \lambda\mathbf{w}$.

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_1 \\ \frac{w_2}{w_1}u_1 \end{pmatrix} = \begin{pmatrix} \lambda w_1 \\ \lambda w_2 \end{pmatrix} = \lambda\mathbf{w}$$

  Now you can solve two equations with one unknowns or you can realize from the first equation that choosing $\lambda = \frac{u_1}{w_1}$ satisfies the equation above.

- $w_2 \neq 0$: Again using $(\star)$ we can write that $u_1 = \frac{w_1}{w_2}u_2$. Now the λ such that $\mathbf{u} = \lambda\mathbf{w}$ must satisfy:

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \frac{w_1}{w_2}u_2 \\ u_2 \end{pmatrix} = \begin{pmatrix} \lambda w_1 \\ \lambda w_2 \end{pmatrix} = \lambda\mathbf{w}$$

  Now you can realize that choosing $\lambda = \frac{u_2}{w_2}$ satisfies the equation above.

- Since we were able to calculate a $\lambda \in \mathbb{R}$ such that $\mathbf{u} = \lambda\mathbf{w}$ in all cases, we conclude that $\mathbf{u} \in L$.

- The subset relation in both ways prove that $L = \{\mathbf{v} \in \mathbb{R}^2 : \mathbf{v} \cdot \mathbf{d} = 0\}$

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