# Linear Algebra
# Why?
# $(A\mathbf{x})^\top = \mathbf{x}^\top A^\top$

G-17

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You are going to talk about the contents of the following in the lecture formally. Here it is mentioned so that you know $(Ax)^\top = (x^\top A^\top)$ actually holds. The definition below did not appear out of nowhere and there is actually logical reasoning behind it. If you feel confused about the contents of this document you can read it again in 1 or 2 weeks.

## 1 Proof

Let $A \in \mathbb{R}^{m\times n}$ and $\mathbf{x} \in \mathbb{R}^n$. To prove the given expression with our knowledge so far, we can start by interpreting $A\mathbf{x}$ in the expression $(A\mathbf{x})^\top$ closer. We are going to use column notation for $A$:

$$A = \begin{bmatrix} - & \mathbf{u}_1^\top & - \\ - & \mathbf{u}_2^\top & - \\ & \vdots & \\ - & \mathbf{u}_m^\top & - \end{bmatrix}$$

where we have $\mathbf{u}_1, \ldots, \mathbf{u}_m \in \mathbb{R}^n$. Now observe the following:

$$(A\mathbf{x})^\top = \left( \begin{bmatrix} - & \mathbf{u}_1^\top & - \\ - & \mathbf{u}_2^\top & - \\ & \vdots & \\ - & \mathbf{u}_m^\top & - \end{bmatrix} \mathbf{x} \right)^\top = \left( \begin{pmatrix} \mathbf{u}_1^\top\mathbf{x} \\ \mathbf{u}_2^\top\mathbf{x} \\ \vdots \\ \mathbf{u}_m^\top\mathbf{x} \end{pmatrix} \right)^\top = \begin{pmatrix} \mathbf{u}_1^\top\mathbf{x} & \mathbf{u}_2^\top\mathbf{x} & \cdots & \mathbf{u}_m^\top\mathbf{x} \end{pmatrix} \in (\mathbb{R}^m)^*$$

Let's keep that result in mind. Now we are going to observe the covector matrix multiplication $x^\top A^\top$. First look at how the factors in our product look like:

$$\mathbf{x}^\top = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}$$

and

$$A^\top = \begin{bmatrix} - & \mathbf{u}_1^\top & - \\ - & \mathbf{u}_2^\top & - \\ & \vdots & \\ - & \mathbf{u}_m^\top & - \end{bmatrix}^\top = \begin{bmatrix} | & | & \cdots & | \\ \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_m \\ | & | & \cdots & | \end{bmatrix} \in \mathbb{R}^{n\times m}$$

Before we go on and multiply these two "objects", we can take a closer look. You can directly use covector-matrix multiplication as defined in the following definition. From the script:

***Definition 2.43*** (*Covector-matrix multiplication*) Let

$$\mathbf{y}^\top = (y_1\ y_2\ \cdots\ y_m) \in (\mathbb{R}^m)^*, \qquad A = \begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ | & | & & | \end{bmatrix} \in \mathbb{R}^{m\times n}.$$

The covector

$$\mathbf{y}^\top A = \underbrace{\begin{pmatrix} \mathbf{y}^\top\mathbf{v}_1 & \mathbf{y}^\top\mathbf{v}_2 & \cdots & \mathbf{y}^\top\mathbf{v}_n \end{pmatrix}}_{n \text{ scalar products}} \in (\mathbb{R}^n)^*$$

is the product of $\mathbf{y}^\top$ and $A$.

So now we have:

$$\mathbf{x}^\top A = \begin{pmatrix} \mathbf{x}^\top\mathbf{u}_1 & \mathbf{x}^\top\mathbf{u}_2 & \cdots & \mathbf{x}^\top\mathbf{u}_n \end{pmatrix}$$

Which gives us the exact result from above if you consider the fact that dot product is commutative (***Observation 1.10 (i)***).

This proves $(A\mathbf{x})^\top = \mathbf{x}^\top A^\top$

$\square$

Though this feels like cheating. We just used a given definition that fits perfectly to our proof. The logic behind this definition is actually matrix-matrix multiplication. You can write

$$\mathbf{x}^\top A^\top = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} | & | & \cdots & | \\ \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_m \\ | & | & \cdots & | \end{bmatrix}$$

You might have already realized that even though $\mathbf{x}$ is a vector, we did not write it with round brackets. Instead it is written in square brackets as a matrix. This is because *everything is a matrix multiplication.* To be more precise, everything can be considered as a matrix-matrix multiplication. In this case you can consider $\mathbf{x} \in \mathbb{R}^{1\times n}$ as a matrix. There is a whole section on why you can make this assumption. (Section 2.3.4) Thus, we can justify that we are actually allowed to go through with the computation above.

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