Linear Algebra
In-Class Exercise Week 6

G-07

04 XI 2024

1. Subspaces of vector spaces (in-class) (★★☆)

a) Let H be a hyperplane of $\mathbb{R}^m$. Recall that this means that there exists a non-zero vector $\mathbf{d} \in \mathbb{R}^m$ with $H = \{\mathbf{v} \cdot \mathbf{d} = 0 : \mathbf{v} \in \mathbb{R}^m\}$. Prove that H is a subspace of $\mathbb{R}^m$.

b) Consider again a hyperplane H of $\mathbb{R}^m$. Prove that the dimension of H is $m - 1$.

1 Solution

The question first asks us to prove H is a subspace of R^m and then determine its dimension. There are patterns for how to solve this type of questions, which are quite often included in linear algebra exams.

If you want to prove that a given subset U of a given space V actually forms a subspace with addition and scalar multiplication as defined in V, you must show 3 things:

1. U is not empty.
2. U is closed under addition.
3. U is closed under scalar multiplication.

And if you want to determine the dimension of a space V, the usual way to calculate it is to construct a basis and show how many vectors are in a basis.

If you construct -or sometimes it is also possible to guess- a basis, then you must also show that your set of vectors actually form a basis by showing the two defining properties *i.e.* basis axioms. Let the set be B:

1. B is a spanning set ($i.e.$ $Span(B) = V$)
2. B is a set of linearly independent vectors.

If you knew the dimension, then one of these properties would imply the other, for example the span of $dim(V)$ linearly independent vectors is the whole vector space V. But here you do not know the dimension od the vetor space so you must prove both basis axioms. Note: We must sometimes also prove $B \subseteq V$ if we construct the vectors ourselves.

The following pages have the necessary steps of the proofs and also some explanations to make it clear. For the solution of subtask c you should look at the solutions on the course website, it provides a quite good explanation.

Solution

a)

Since H is a hyperplane there exists a non-zero vector $\mathbf{d}$ that is orthogonal to all vectors in H. This is the definition of a hyperplane. So for some $\mathbf{d}\ in\mathbb{R}^m$

$$H = \{\mathbf{v} \in \mathbb{R}^m | \ \mathbf{v} \cdot \mathbf{d} = 0\}$$

We prove 3 things mentioned above:

1. By definition $\mathbf{0} \in H$ so H is not empty.

   Now let $\mathbf{v}, \mathbf{u} \in H$ and $c \in \mathbb{R}$ be arbitrary. We have to show $(\mathbf{v}+\mathbf{u}) \in H$ and $c\mathbf{v} \in H$.
2. $(\mathbf{v} + \mathbf{u}) \cdot \mathbf{d} = \mathbf{v} \cdot \mathbf{d} + \mathbf{u} \cdot \mathbf{d} = 0 + 0 = 0$ and therefore $(\mathbf{v} + \mathbf{u}) \in H$.
3. $(c\mathbf{v}) \cdot \mathbf{d} = c(\mathbf{v} \cdot \mathbf{d}) = c0 = 0$, so we have $c\mathbf{v} \in H$.

And thereby we have proven that H forms a subspace of $\mathbb{R}^m$

b)

1.1 Choosing a basis

We go with the general pattern here: Find a basis use it as a proof for the dimension. In some cases we can guess. The master solution suggests:

Let $H = \{\mathbf{v} \in \mathbb{R}^m : \mathbf{v} \cdot \mathbf{d} = 0\}$ for some non-zero $\mathbf{d} \in \mathbb{R}^m$. Consider the standard unit vectors $\mathbf{e}_1, \dots, \mathbf{e}_m \in \mathbb{R}^m$ and let $d_i := \mathbf{e}_i \cdot \mathbf{d}$ for all $i \in [m]$. Observe that we must have $d_j \neq 0$ for some $j \in [m]$, because $\mathbf{d}$ is non-zero. For every $i \in [m]$, define the vector $\mathbf{v}_i := \mathbf{e}_i - \frac{d_i}{d_j}\mathbf{e}_j$. We claim that the set of vectors $\{\mathbf{v}_i : i \neq j\}$ is a basis of H.

As an overview the suggested basis is the following set of $(m-1)$vectors:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ -\frac{d_1}{d_j} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ -\frac{d_2}{d_j} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \\ -\frac{d_3}{d_j} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \dots, \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ -\frac{d_m}{d_j} \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

This would be an incredibly lucky guess. So how do we arrive the conclusion that this might actually be a basis? Let's start over.

All information we have about the vectors $\mathbf{v}$ in our subspace H is the following property: There exists a $\mathbf{d} \in \mathbb{R}^m$ such that $\mathbf{v} \cdot \mathbf{d} = 0$ If we write this in more detail, the scalar product corresponds to:

$$v_1 d_1 + v_2 d_2 + \dots + v_m d_m = 0$$

We have to choose $\mathbf{v}$ such that its elements satisfy this equality for some *fixed* $\mathbf{d}$. We want to choose so many different vectors $\mathbf{v}$ as we can, which are going to be our basis. Therefore we should also be careful that they are linearly independent. So, finding a solution, a vector satisfying the equation and then doubling all its elements would not work. Since we are free to choose so far as we satisfy the basis requirements, we can as well have many 0 entries in $\mathbf{v}$. Let v_1 be 1. Then our equation from above looks like:

$$d_1 + \cdots = 0$$

We apparently want " . . . " to be equal to $-d_1$. So for some v_j and some non-zero d_j, which always exists because $\mathbf{d} \neq \mathbf{0}$ by the definition of a hyperplane, we should have $v_j = -\frac{d_1}{d_j}$ so that in the end our equation is

$$1 \cdot d_1 + \cdots - \frac{d_1}{d_j} \cdot d_j + \cdots = 0$$

See that we have $d_1 - d_1 = 0$ if you do not consider the " . . . " So we choose all other elements of $\mathbf{v}$ except for v_1 and v_j as all 0's. Voilà, you have a vector $\mathbf{v}$ that satisfies $\mathbf{v} \cdot \mathbf{d} = 0$.

If you fix your the index j above, s.t. $d_j \neq 0$ and do this for every v_i except $i \neq j$, you have $(m - 1)$ vectors that are all solutions for $\mathbf{v} \cdot \mathbf{d} = 0$. Why $i \neq j$? Because then you would have to choose another index to negate $v_j d_j$, through the same process as above. But then the vector you had would be linearly dependent to other vectors you have. You can try this on a smaller, concrete example. In other words we can say that we only have $m - 1$ free variables in our equation.

Now we have a candidate for a basis that consists of $(m - 1)$ vectors, where the i-th vector $\mathbf{v}_i = \mathbf{e}_i - \frac{d_i}{d_j}\mathbf{e}_j$, same as suggested in the master solution. More formally our candidate basis is the set of vectors $B = \{\mathbf{v}_i \mid i \in [m], i \neq j\}$

1.2 Showing that our chosen set of vectors is actually a basis

Only thing left to prove is that B is actually a basis of H. We have to show three things: Even though we ourselves intuitively know all $\mathbf{v}_i \in H$ since we constructed the vectors ourselves intuition is not enough and we nevertheless have to show that all vectors $B \subseteq H$. Secondly, we need to show $\mathbf{v}_i$ are linearly independent for all $i \in [m], i \neq j$ and thirdly, $Span(B) = H$, which means every vector in H can be written as a linear combination of vectors in B.

- $B \subseteq H$**:** There is an easy way to prove this. We have to show $\mathbf{v}_i\mathbf{d} = 0$ for all $\mathbf{v}_i \in B$. See that:

$$\mathbf{v}_i\mathbf{d} = (\mathbf{e}_i - \frac{d_i}{d_j}\mathbf{e}_j)\mathbf{d} = \mathbf{e}_i\mathbf{d} - \frac{d_i}{d_j}\mathbf{e}_j\mathbf{d} = d_i - d_i = 0$$

  Thus $B \subseteq H$.

- **Linear Independence:** You can argue with the private non-zero argument. All vectors $\mathbf{v}_i$ in B control their own coordinate, so it is not possible to write $\mathbf{0}$ as a nontrivial linear combination of the vectors in B. They are linearly independent.

- **Spanning H:** Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} \in H$ We know that $\mathbf{u} \cdot \mathbf{d} = 0$. This means we have $u_1d_1 + u_2d_2 + \cdots + u_md_m = 0$ Which means for our j from above we have $u_1d_1 + u_2d_2 + \cdots + u_{j-1}d_{j-1} + u_{j+1}d_{j+1} + \cdots + u_md_m = -u_jd_j$ In compact form:

$$\sum_{\substack{i=1 \\ i \neq j}}^{m} u_id_i = -u_jd_j$$

  Having this in mind, we claim that we can write $\mathbf{u}$ as a linear combination of $\mathbf{v}_i$'s if we multiply the vectors $\mathbf{v}_i$ with the elements of $\mathbf{u}$. We show this in the following:

$$\begin{aligned} \sum_{i \in [m] \setminus \{j\}} u_i\mathbf{v}_i &= \sum_{i \in [m] \setminus \{j\}} u_i \left(\mathbf{e}_i - \frac{d_i}{d_j}\mathbf{e}_j\right) \\ &= \sum_{i \in [m] \setminus \{j\}} u_i\mathbf{e}_i - \left(\sum_{i \in [m] \setminus \{j\}} u_id_i\right) \frac{1}{d_j}\mathbf{e}_j \\ &= (\sum_{i \in [m] \setminus \{j\}} u_i\mathbf{e}_i) + u_j\mathbf{e}_j \\ &= \mathbf{u} \end{aligned}$$

  Since $\mathbf{u}$ was arbitrary, this shows for any $\mathbf{u} \in H$ that $\mathbf{u} \in Span(B)$. Or in other words: $Span(B) = H$

All these 3 steps together prove that B is a basis of H.
Since there are $m-1$ vectors in a basis of H, we conclude that
$dim(H) = m-1$.

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